Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 11

All exercises on this sheet are counted as ***bonus exercises****. The final exam will contain* ***eight assignments****, which resemble the exercises on this sheet in extent and difficulty, but may involve different topics.*

Exercise 1. (a) Let $(X_i)_{i\in I}$ be a family of integrable r.v.'s on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with $\sup_{i\in I} \mathbb{E}|X_i| < \infty$. Show that $\{\mathbb{P}_{X_i} : i \in I\}$ is uniformly tight.

(b) Let $I \subseteq (0,\infty)$. Show that the family $\mathcal{P}$ of exponential distributions on $\mathbb{R}$ with $\mathcal{P} := \{\mathfrak{E}xp(\alpha) : \alpha \in I\}$ is uniformly tight if and only if $I \subseteq [K,\infty)$ for some $K > 0$.

(2* + 2* points)

Exercise 2. Let $(N_t)_{t\geqslant 0}$ be a Poisson process on $(\Omega, \mathscr{A}, \mathbb{P})$ of intensity $\lambda > 0$ with jump times $(S_k)_{k\in\mathbb{N}}$. Show that for all $k \in \mathbb{N}$, $t > 0$, $u \in (0, t]$ and $v > 0$ it holds that:

$$\mathbb{P}(t-u < S_k \leqslant t, t < S_{k+1} \leqslant t+v) = \frac{(\lambda t)^k - (\lambda(t-u))^k}{k!} e^{-\lambda t}(1-e^{-\lambda v})$$

Hint: Use without proof that for i.i.d. $\mathfrak{E}xp(\lambda)$*-distributed r.v.'s* $\varepsilon_1, ..., \varepsilon_k$ *the sum* $U_k := \sum_{i=1}^n \varepsilon_i$ *has a* $\Gamma(k,\lambda)$*-distribution with Lebesgue density* $f_{U_k}(x) = \frac{\lambda^k}{(k-1)!} x^{k-1} e^{-\lambda x} \mathbb{1}_{[0,\infty)}(x)$*. Thus, infer the joint density* $f_{S_k, S_{k+1}-S_k}(x,y)$ *of* $(S_k, S_{k+1} - S_k)$*.*

(4* points)

Exercise 3. Let $(\varepsilon_k)_{k\in\mathbb{N}_0}$ be an i.i.d. sequence of r.v.'s with distribution $\mathfrak{N}(0,\sigma^2)$, $\sigma > 0$. Define the processes $X_n := \left(\sum_{k=n}^{n+3} \varepsilon_k\right)^2$ and $Y_n := \sum_{k=3n}^{4n} \varepsilon_k$.

(a) Show that $(X_n)_{n\in\mathbb{N}_0}$ is stationary and ergodic while $(Y_n)_{n\in\mathbb{N}_0}$ is *not* stationary.

(b) Show that $\frac{1}{n}\sum_{i=0}^{n-1} X_i$ converges a.s. and find the limit.

(2*+2* points)

Exercise 4. Let $(B_t)_{t\geqslant 0}$ be a Brownian motion on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Fix real numbers $0 \leqslant a < b$.

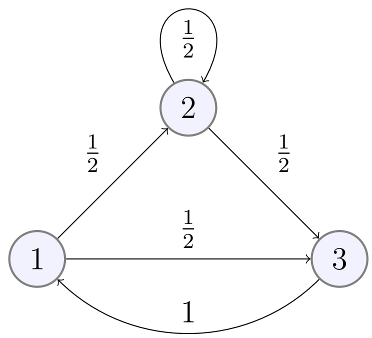
(a) Show that the set $\{\omega \in \Omega : B_t(\omega)$ is monotone increasing on $[a,b]\}$ is $\mathscr{A}$-measurable.

(b) For $n \in \mathbb{N}$, calculate the probability $\mathbb{P}(B_a \leqslant B_{a+\frac{b-a}{n}} \leqslant ... \leqslant B_b)$.

(c) By considering the limit $n \to \infty$, conclude that $\mathbb{P}$-a.s. (B_t) is not monotone increasing on the interval $[a,b]$.

(1*+2*+1* points)

Exercise 5. Let $(X_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain on the finite state space $\mathcal{S} := \{1, 2, 3\}$. The transition probabilities $p_{i,j}$, $i, j \in \mathcal{S}$ are given by the following graph:



(a) Calculate $\mathbb{P}_2(X_2 = 3)$.

(b) Show that $(X_n)_{n\in\mathbb{N}_0}$ is irreducible.

(c) Classify the states in $\mathcal{S}$ into recurrent and transient states.

(d) Calculate a stationary distribution of the Markov chain. Is it unique?

(1*+1*+1*+1* points)

Exercise 6. A Brownian bridge $(Z_t)_{t\in[0,1]}$ is a continuous, centred Gaussian process with covariance function $c(t, s) = t \wedge s - st$, $s, t \in [0, 1]$. Let $(B_t)_{t\geqslant 0}$ be a Brownian motion.

(a) Show that the process $X_t := B_t - tB_1$, $t \in [0, 1]$, is a Brownian bridge.

(b) Show that the process $\widetilde{X}_t = (1 - t)B_{\frac{t}{1-t}}$, $t \in [0, 1)$, $\widetilde{X}_1 = 0$, is a Brownian bridge.

(2*+2* points)

Handing in **voluntarily** during the lecture on Friday, 15 July 2016