Lecture course *Probability Theory II*
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Exercise sheet 10

Exercise 1. Show that for a sequence of probability measures $(\mathbb{P}_n)_{n\in\mathbb{N}}$ on $\mathcal{C}([0,1])$:

$$\forall \varepsilon > 0 : \lim_{\delta\to 0} \limsup_{n\to\infty} \sup_{t\in[0,1-\delta]} \delta^{-1}\mathbb{P}_n\left(\max_{s\in[t,t+\delta]} |f(s)-f(t)| \geqslant \varepsilon\right) = 0$$

implies for the modulus of continuity $w_f(\delta)$

$$\forall \varepsilon > 0 : \lim_{\delta\to 0} \limsup_{n\to\infty} \mathbb{P}_n(w_f(\delta) \geqslant \varepsilon) = 0.$$

(4 points)

Exercise 2. Let $(\mathcal{S}, d)$ be a metric space and $\mathscr{B}(\mathcal{S})$ the Borel-σ-algebra of $\mathcal{S}$. For $m, n \in \mathbb{N}$ let $T_n, T_{n,m}, T$ and T_m be r.v.'s with values in $(\mathcal{S}, \mathscr{B}(\mathcal{S}))$. Show that $T_n \xrightarrow{d} T$ as $n \to \infty$, if the following conditions hold:

(a) $\forall m \in \mathbb{N}$: $T_{n,m} \xrightarrow{d} T_m$ as $n \to \infty$.

(b) $T_m \xrightarrow{d} T$ as $m \to \infty$.

(c) $\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}(d(T_{n,m}, T_n) > \varepsilon) = 0$ for all $\varepsilon > 0$.

Hint: Use the Portemanteau theorem, §6.1.4 (v). For a closed set $F \subseteq \mathcal{S}$ *define* $d(x, F) := \inf\{d(x,y), y \in F\}$, $\overline{B}_\varepsilon(F) := \{x \in \mathcal{S} : d(x,F) \leqslant \varepsilon\}$ *and for r.v.'s* X, Z *use that* $\mathbb{P}(Z \in F) \leqslant \mathbb{P}(d(X,Z) > \varepsilon) + \mathbb{P}(X \in \overline{B}_\varepsilon(F))$.

(4 points)

Exercise 3. Let $(W_t)_{t\geqslant 0}$ be a Brownian motion as defined in §2.1.7 on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. For an interval $[a,b] \subseteq [0,\infty)$ define the variation

$$V_{[a,b]}(W) := \sup_{n\in\mathbb{N}, a=t_0<\ldots<t_n=b} \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|.$$

(a) Let $t_i := a + \frac{i}{n}(b-a)$, $i \in \{0, \ldots, n\}$ be a uniform partition of the interval $[a,b]$, $n \in \mathbb{N}$. Show that

$$\sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^2 \xrightarrow{\mathbb{P}} b - a \quad \text{as } n \to \infty.$$

(b) Define $M^*(n) := \max_{i=1,\dots,n} |W_{t_i} - W_{t_{i-1}}|$. Show that

$$V_{[a,b]}(W) \geqslant \frac{1}{M^*(n)} \sum_{i=1}^{n} |W_{t_i} - W_{t_{i-1}}|^2,$$

conclude that $(W_t)_{t\in[a,b]}$ has unbounded variation on $[a,b]$ a.s.: $V_{[a,b]}(W) = \infty$ a.s.
Hint: For a sequence of real r.v.'s $(X_n)_{n\in\mathbb{N}}$, the following characterisation holds: $X_n \xrightarrow{\mathbb{P}} X$ as $n \to \infty$ if and only if for every subsequence $(X_{n_k})_{k\in\mathbb{N}}$ there exists a further subsequence $(X_{n_{k_l}})_{l\in\mathbb{N}}$ such $X_{n_{k_l}} \xrightarrow{a.s.} X$ as $l \to \infty$.

(4 points)

Exercise 4. Let $(W_t)_{t\geqslant 0}$ be a Brownian motion, $\mu \in \mathbb{R}$ and $\sigma > 0$. Define the r.v.

$$S_t = \exp\left(\mu t + \sigma W_t\right).$$

(a) Calculate the mean and the variance of S_t for $t \geqslant 0$.

(b) Calculate the probability density function of S_t for $t > 0$.

(c) Under which condition is S_t a martingale with respect to the canonical filtration generated by the Brownian motion?

(4 points)