Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 9

Exercise 1. Consider $(\mathbb{R}, |\cdot|)$ equipped with its Borel-σ-algebra $\mathscr{B}$. Let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence in $(\mathbb{R}, |\cdot|)$ with limit x. Show directly without using the Portemanteau theorem that

(a) $\delta_x = w\text{-}\lim_{n\to\infty} \delta_{x_n}$.

(b) $\liminf\limits_{n\to\infty} \delta_{x_n}(O) \geqslant \delta_x(O)$ for all open $O \subseteq \mathbb{R}$.

(c) $\delta_{x_n}(A) \overset{n\to\infty}{\longrightarrow} \delta_x(A)$ for all $A \in \mathscr{B}$ with $\delta_x(\partial A) = 0$, ∂A the boundary of A. (4 points)

Exercise 2. Let $(\mathcal{S}, d)$ be a metric space equipped with Borel-σ-algebra. For $n \in \mathbb{N}$ let X_n, X, Y_n, Y be $\mathcal{S}$-valued r.v.'s defined on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Assume that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$.

(a) Give a counterexample to show that in general, one can not conclude $(X_n, Y_n) \xrightarrow{d} (X, Y)$ in $\mathcal{S} \times \mathcal{S}$ endowed with the product topology.

(b) Assume that for some $c \in \mathcal{S}$ it holds that $\mathbb{P}(X = c) = 1$. Then $X_n \xrightarrow{\mathbb{P}} X$.

(c) Under the assumption of (b), show that $(X_n, Y_n) \xrightarrow{d} (X, Y)$.

Hint: Use the Portemanteau theorem for (b) and (c). (4 points)

Exercise 3. Let $X, X_1, X_2, \dots$ be $\mathbb{Z}$-valued r.v.'s on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$. Show that $X_n \xrightarrow{d} X$ if and only if $\mathbb{P}(X_n = m) \overset{n\to\infty}{\longrightarrow} \mathbb{P}(X = m)$ for all $m \in \mathbb{Z}$. (4 points)

Exercise 4. (a) Consider $([0, 1], |\cdot|)$ equipped with its Borel-σ-algebra $\mathscr{B}$. Show that $\frac{1}{n}\sum_{k=1}^{n} \delta_{k/n} \xrightarrow{w} \lambda$, where λ is the Lebesgue measure on $\mathbb{R}$ restricted to $[0, 1]$.

(b) For each $n \in \mathbb{N}$ let X_n be a geometrically distributed r.v. with parameter $p_n \in (0, 1)$. How must we choose the sequence $(p_n)_{n\in\mathbb{N}}$ in order that $\mathbb{P}_{X_n/n}$ converges weakly to the exponential distribution with parameter $\alpha > 0$? (4 points)

Handing in during the lecture on Friday, 01 July 2016