Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

Prof. Dr. Jan JOHANNES

Exercise sheet 8

Exercise 1. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space and G a finite group of measure-preserving measurable maps $g : \Omega \to \Omega$. Let $X \in L_1(\Omega, \mathscr{A}, \mathbb{P})$ and $\mathscr{A}_0 := \{A \in \mathscr{A} : g^{-1}(A) = A \text{ for all } g \in G\}$.

(a) Show that $\mathscr{A}_0$ is a σ-Algebra and $\sum_{g \in G} X \circ g$ is $\mathscr{A}_0$-measurable.

(b) Prove

$$\mathbb{E}(X|\mathscr{A}_0) = \frac{1}{\#G} \sum_{g \in G} X \circ g.$$

(4 points)

Exercise 2. Consider the probability space $([0,1), \mathscr{B}([0,1)), \lambda)$ and for $r \in (0,1)$ the rotation $T_r(x) = x + r \text{ mod } 1$. We will show that the measure-preserving dynamical system $([0,1), \mathscr{B}([0,1)), \lambda, T_r)$ is ergodic if and only if r is irrational.

(a) Let $f : [0,1) \to \mathbb{R}$ be $\mathscr{I}_{T_r}$-measurable and (without loss of generality) bounded and thus square-integrable. Expand f in a Fourier series and use the condition $f = f \circ T_r$ from \$5.1.9 to show that f is a.s. constant if $r \in \mathbb{R} \setminus \mathbb{Q}$.

(b) For $r \in \mathbb{Q}$ construct a map that is $\mathscr{I}_{T_r}$-measurable but not a.s. constant.

(4 points)

Exercise 3. A measure-preserving dynamical system $(\Omega, \mathscr{A}, \mathbb{P}, T)$ is called (strongly) mixing if

$$\lim_{n\to\infty} \mathbb{P}(A \cap T^{-n}(B)) = \mathbb{P}(A)\mathbb{P}(B)$$

holds for all $A, B \in \mathscr{A}$.

(a) Show that a (strongly) mixing dynamical system is ergodic, but the converse is in general false.

(b) Prove that $(\Omega, \mathscr{A}, \mathbb{P}, T)$ is ergodic if and only if for all $A, B \in \mathscr{A}$ it holds that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mathbb{P}(A \cap T^{-1}(B)) = \mathbb{P}(A)\mathbb{P}(B).$$

Hint: Apply Birkhoff's ergodic theorem to $\mathbb{1}_B$ for one direction.

(4 points)

Exercise 4. Consider the probability space $([0,1), \mathscr{B}([0,1)), \lambda)$.

(a) Show that the shift function $T : [0,1) \to [0,1)$, $x \mapsto 2x \bmod 1$ is measurable, measure preserving and ergodic.

(b) Apply the ergodic theorem to the function $\mathbb{1}_{[1/2,1)}$ to conclude that for any $x \in [0,1)$ the proportion of 1's in the binary representation of x is a.s. $1/2$.

(4 points)
