Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 7

Exercise 1. Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain on an at most countable state space $(\mathcal{S}, \mathscr{S})$ with transition matrix P.

(a) Show that for $i \in \mathcal{S}$ positive recurrent and $\rho_{ij} > 0$, also j is positive recurrent.

(b) Suppose that $\psi : \mathcal{S} \to \mathbb{R}$ is bounded and satisfies $\sum_{j\in\mathcal{S}} P_{ij}\psi(j) \leqslant \lambda\psi(i)$ for some $\lambda > 0$ and all $i \in \mathcal{S}$. Show that $\lambda^{-n}\psi(X_n)$ constitutes a supermartingale. (4 points)

Exercise 2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain on the state space $\mathcal{S} = \{0, 1\}$ with transition matrix

$$p = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix},$$

where $\alpha + \beta > 0$.

(a) Calculate the correlation $\rho(X_m, X_{m+n})$, and its limit for $m \to \infty$ for fixed n.

(b) Find $\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \mathbb{P}(X_i = 1)$.

(c) Under which conditions is the process stationary? (4 points)

Exercise 3. (a) Characterise the measure preserving transformations on a finite set $\Omega = \{\omega_1, ..., \omega_n\}$ with $n \geqslant 1$ elements, given a probability measure $\mathbb{P}$ on the power set 2^Ω.

(b) Assume now that $\mathbb{P}(\{\omega_i\}) > 0$ for all $1 \leqslant i \leqslant n$. Show that there exists a unique probability measure $\mathbb{P}$ on the power set, such that there is an ergodic transformation $T : \Omega \to \Omega$. How do $\mathbb{P}$ and T look like? (4 points)

Exercise 4. Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space. Prove or disprove:

(a) Let $(U_n)_{n\in\mathbb{N}_0}$ and $(V_n)_{n\in\mathbb{N}_0}$ be real-valued stationary processes on $(\Omega, \mathscr{A}, \mathbb{P})$. Then the process $(W_n)_{n\in\mathbb{N}_0}$ with $W_n = U_n V_n$ is stationary.

(b) Every stationary process $(Z_n)_{n\in\mathbb{N}_0}$ on $(\Omega, \mathscr{A}, \mathbb{P})$ is a martingale with respect to the natural filtration $\mathscr{F}^Z$.

(c) Let $(X_n)_{n\in\mathbb{N}_0}$ be a real-valued stationary process. Then the process $(Y_n)_{n\in\mathbb{N}_0}$ with

$$Y_n = \sum_{i=0}^{\infty} (-1)^i \frac{(X_n + X_{n+1})^{2i+1}}{(2i+1)!}$$

is again stationary.

(d) For $A \in \mathbb{R}_+$, $\varphi \in (0, 2\pi)$ and U having a uniform distribution in $[0, 2\pi]$, the sequence $(X_t)_{t\in\mathbb{Z}}$ where $X_t = A\cos(\phi t + U)$ is stationary.

(4 points)

Handing in during the lecture on Friday, 17 June 2016