Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 6

Exercise 1. Let $(S_n)_{n\in\mathbb{N}_0}$ be a one-dimensional symmetric simple random walk, i.e. $S_n = \sum_{i=1}^n X_i$, with i.i.d. r.v.'s (X_i) satisfying $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = 1/2$.

(a) Calculate the Doob decomposition of the process $|S| = (|S_n|)_{n\in\mathbb{N}_0}$.

(b) Calculate the expected displacement from the origin $\mathbb{E}(|S_n|)$ at time $n \in \mathbb{N}_0$.
Hint: Show first that $\mathbb{E}(|S_n|) = \mathbb{E}(\#\{i \leqslant n-1 : S_i = 0\})$. (4 points)

Exercise 2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a square integrable martingale with square variation process $\langle X\rangle$. Let τ be a finite stopping time. Show the following:

(a) If $\mathbb{E}(\langle X\rangle_\tau) < \infty$, then $\mathbb{E}((X_\tau - X_0)^2) = \mathbb{E}(\langle X\rangle_\tau)$ and $\mathbb{E}(X_\tau) = \mathbb{E}(X_0)$.

(b) Both equalities in (a) can fail if $\mathbb{E}(\langle X\rangle_\tau) = \infty$. (4 points)

Exercise 3. (a) Let $X = (X_t)_{t\in\mathbb{T}}$ have the Markov property with respect to the natural filtration $\mathscr{F}$, that is, for every $A \in \mathscr{S}$ and all $s, t \in \mathbb{T}$ with $s \leqslant t$, $\mathbb{P}(X_t \in A|\mathscr{F}_s) = \mathbb{P}(X_t \in A|X_s)$. Show that the distribution of X is uniquely determined by its one- and two-dimensional marginal distributions.

(b) Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a time-homogeneous Markov chain with finite state space $\mathcal{S}$ and transition probabilities $(P_{ij})_{i,j\in\mathcal{S}}$. Show that there exists at least one recurrent state. (4 points)

Exercise 4. Let $(X_n)_{n\in\mathbb{N}_0}$ be a Markov chain with arbitrary initial distribution and transition matrix

$$P = \begin{pmatrix} 1/3 & 0 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1/3 & 1/3 & 0 & 1/3 & 0 & 0 \end{pmatrix}$$

Draw a graph representing the transition probabilities with states as vertices and edges for transitions to clarify the structure of the Markov chain. Find all recurrent, transient, closed and irreducible sets of states. (4 points)

Handing in during the lecture on Friday, 10 June 2016