Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 5

Exercise 1. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. real-valued r.v.'s in L_2 defined on a filtered probability space $(\Omega, \mathscr{A}, \mathbb{P}, \mathscr{F}^X)$ with natural filtration $\mathscr{F}^X$.

(a) Show that the process $((S_n - n\mathbb{E}X_1)^2 - n\,\mathbb{V}\mathrm{ar}\,X_1)_{n\in\mathbb{N}}$ with $S_n := \sum_{i=1}^n X_i$, $n \in \mathbb{N}$, is an integrable martingale.

(b) Prove that every stopping time τ with $\mathbb{E}(\tau) < \infty$ is regular for $((S_n - n\mathbb{E}X_1)^2 - n\,\mathbb{V}\mathrm{ar}\,X_1)_{n\in\mathbb{N}}$ and that it satisfies *Wald's identity* $\mathbb{E}[S_\tau - \tau\mathbb{E}(X_1)]^2 = \mathbb{E}(\tau)\,\mathbb{V}\mathrm{ar}(X_1)$.

(c) If in addition $\mathbb{E}(\tau^2) < \infty$ conclude that $\mathbb{V}\mathrm{ar}(S_\tau) = \mathbb{V}\mathrm{ar}(\tau)(\mathbb{E}X_1)^2 + \mathbb{E}(\tau)\,\mathbb{V}\mathrm{ar}(X_1)$.

(4 points)

Exercise 2. In each round of a game a fair coin is tossed. For heads the player receives his double stake, for tails he loses his stake. The initial capital of the player is $K_0 = 0$. At game $n > 1$ his strategy is as follows: If heads has appeared before, his stake is zero (he stops playing); otherwise his stake is 2^{n-1} Euro.

(a) Argue why his capital K_n after game n can be modelled with independent r.v.'s $(X_i)_{i\in\mathbb{N}}$ such that $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = 1/2$ via

$$K_n = \begin{cases} -(2^n - 1), & X_1 = X_2 = \ldots = X_n = -1, \\ 1, & \text{otherwise} \end{cases}$$

(b) Represent K_n as martingale transform.

(c) Prove $\lim_{n\to\infty} K_n = 1$ a.s. although $\mathbb{E}(K_n) = 0$ for all $n > 0$ holds. (4 points)

Exercise 3. A particle with radius r is immersed in a fluid with dynamic viscosity η. We assume that its trajectory is a two-dimensional, real-valued random process that can be described by two independent Brownian motions $(W_{1,t})_{t\geqslant 0}$ and $(W_{2,t})_{t\geqslant 0}$ (see $ 2.1.7), scaled with a factor $\sqrt{D} > 0$, i.e. $\widetilde{W}_{i,t} = \sqrt{D}W_{i,t}$. The mean squared deviation from the initial position is related to the time t by

$$\mathbb{E}(\widetilde{W}_{t,1}^2 + \widetilde{W}_{t,2}^2) = 2Dt, \qquad D = \frac{k_B T}{6\pi\eta r},$$

where T is the temperature and k_B is called *Boltzmann constant*.

(a) Show that $(\widetilde{W}_{1,t}^2 + \widetilde{W}_{2,t}^2 - 2Dt)_{t \geqslant 0}$ is a martingale.

(b) Find the mean time that passes until the centre of the particle leaves the disk $B_a(0)$, $a > 0$. Assume in an experiment r, η, a and T are known. How can one determine the Boltzmann constant from the mean exit time above?
Hint: Define the stopping time $\tau = \inf\{t \geqslant 0 : \widetilde{W}_{1,t}^2 + \widetilde{W}_{2,t}^2 \geqslant a^2\}$ *and use the optional stopping theorem for the martingale from (a).* (4 points)

Exercise 4. In a hallway of a hotel there are K rooms and K people who make an attempt to find their room by picking one at random (each room belongs to exactly one person). Those who find their own room leave the hallway, the rest try again. Let N be the number of rounds of attempts until the hallway is empty. Show that $\mathbb{E}(N) = K$ and $\mathbb{V}ar(N) \leqslant K$.
Hint: Let M_n *be the number of people in the hallway after round* n. *Show that* $(M_n+n)_{n\in\mathbb{N}_0}$ *is a martingale and* $((M_n + n)^2 + M_n)_{n\in\mathbb{N}_0}$ *is a supermartingale with respect to a suitable filtration.* (4 points)

Handing in during the lecture on Friday, 03 June 2016

Written Exam on Wednesday, 27 July 2016, 11 a.m. to 1 p.m.