Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

Prof. Dr. Jan JOHANNES

Exercise sheet 4

Exercise 1. A function $f \in C^2(\mathbb{R}^d, \mathbb{R})$ is called superharmonic, if for all $x \in \mathbb{R}^d$, $\Delta f(x) = \sum_{i=1}^d \frac{\partial^2 f}{\partial x_i^2}(x) \leqslant 0$. In this case the following inequality holds for all balls $B_r(x) \subseteq \mathbb{R}^d$, $r > 0$, $x \in \mathbb{R}^d$ (no proof required):

$$f(x) \geqslant \frac{1}{\lambda_d(B_r(0))} \int_{B_r(x)} f(y) d\lambda_d(y). \tag{1}$$

(a) For a measurable function $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $\mathbb{R}^d$-valued independent r.v.'s X, Y with $f(X,Y) \in L_1$ show first that there holds $\mathbb{E}[f(X,Y)|Y] = g(Y)$ with $g(y) = \mathbb{E}(f(X,y))$.

(b) Conclude that for a superharmonic function f on $\mathbb{R}^d$ and $(\xi_i)_{i\geqslant 0}$ i.i.d. uniform r.v.'s on $B_1(0)$, $f(S_n)$ is a $(\sigma(\xi_1, ..., \xi_n))_{n\geqslant 1}$-supermartingale, where $S_n = x + \sum_{i=1}^n \xi_i$.

(c) In the case $d = 1$, prove the assertion from (b) without using (1). (4 points)

Exercise 2. (a) Let $\mathscr{F}$ be a filtration and $(X_n)_{n\in\mathbb{N}}$ a positive $\mathscr{F}$-supermartingale. Let $0 < a < b < \infty$ and $\beta_{a,b}$ the number of upcrossings associated with $(X_n)_{n\in\mathbb{N}}$. Use Dubin's inequality (§3.1.11) to prove the inequality

$$\mathbb{E}\beta_{a,b} \leqslant \frac{a}{b-a} \mathbb{E} \min\left(\frac{X_1}{a}, 1\right). \tag{2}$$

(b) Prove that if a positive supermartingale $(X_t)_{t\in\mathbb{N}}$ attains the value zero, then it is zero for all times afterwards.
Hint: Use Dubins' equality in the limit $a \to 0$ to show that with probability 1, there is no upcrossing of the interval $[0, b)$ for any $b > 0$. (4 points)

Exercise 3. Give a proof of Kolmogorov's 0-1-law based on martingale theory:

(a) Let $(\mathscr{F}_n)_{n\in\mathbb{N}}$ be a filtration and $\mathscr{F}_\infty := \bigvee_{n\in\mathbb{N}} \mathscr{F}_n$. Show that for $A \in \mathscr{F}_\infty$, we have $\mathbb{E}[\mathbb{1}_A|\mathscr{F}_n] \xrightarrow{a.s.} \mathbb{1}_A$ as $n \to \infty$.

(b) Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of independent r.v.'s. Consider the natural filtration $(\mathscr{F}_n)_{n\in\mathbb{N}}$ with $\mathscr{F}_n = \sigma(X_k, k \leqslant n)$ and the tail σ-algebra $\mathscr{G} = \bigcap_{n\in\mathbb{N}} \sigma(X_k, k \geqslant n)$.

For $A \in \mathscr{G}$, deduce that $\mathbb{P}(A) = \mathbb{E}[\mathbb{1}_A|\mathscr{F}_n] \xrightarrow{a.s.} \mathbb{1}_A$ such that $\mathbb{P}(A) \in \{0, 1\}$ holds.
(4 points)

Exercise 4. Let $(X_n)_{n\in\mathbb{N}}$ be an integrable {super/sub-}martingale. Show that for every stopping time τ, the stopped process $X^\tau = (X_n^\tau)_{n\in\mathbb{N}}$, defined by $X_n^\tau = X_{\tau\wedge n}$ for any $n \in \mathbb{N}$ is again an integrable {super/sub-}martingale. (4 points)