Lecture course *Probability Theory II*
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Exercise sheet 3

Exercise 1. Let $C([0,\infty))$ be equipped with the topology of uniform convergence on compacts using the metric $d(f,g) := \sum_{k\geqslant 1} 2^{-k}\left(\sup_{t\in[0,k]}|f(t)-g(t)|\wedge 1\right)$. Prove

(a) $(C([0,\infty)), d)$ is Polish.

(b) The Borel σ-algebra is the smallest σ-algebra that makes all coordinate projections $\pi_t : C([0,\infty)) \to \mathbb{R}$, $t \geqslant 0$, measurable.

(c) For any continuous stochastic process $(X_t)_{t\geqslant 0}$ on $(\Omega, \mathscr{A}, \mathbb{P})$ the mapping $\overline{X} : \Omega \to C([0,\infty))$ with $\overline{X}(\omega)_t := X_t(\omega)$ is Borel-measurable.

(d) The law of $\overline{X}$ is uniquely determined by the finite-dimensional distributions of X.

(4 points)

Exercise 2. A Gaussian process $(X_t)_{t\in\mathbb{T}}$ is a process with (generalized) Gaussian finite-dimensional distributions: For any $t_1,\dots,t_n \in \mathbb{T}$, $(X_{t_1},\dots,X_{t_n}) \sim \mathfrak{N}(\mu_{t_1,\dots,t_n}, \Sigma_{t_1,\dots,t_n})$, where $\mu_{t_1,\dots,t_n} \in \mathbb{R}^n$ and $\Sigma_{t_1,\dots,t_n} \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite.

(a) Argue that the finite-dimensional distributions of a Gaussian process $(X_t)_{t\in\mathbb{T}}$ are uniquely determined by the expectation function $t \mapsto \mathbb{E}(X_t)$ and the covariance function $(s,t) \mapsto \mathbb{C}\mathrm{ov}(X_s, X_t)$.

(b) Show that for any function $\mu : \mathbb{T} \to \mathbb{R}$ and any symmetric, positive semi-definite function $C : \mathbb{T}^2 \to \mathbb{R}$, i.e. $C(t,s) = C(s,t)$ and

$$\forall n \geqslant 1; t_1,\dots,t_n \in \mathbb{T}; \lambda_1,\dots,\lambda_n \in \mathbb{R} \; : \; \sum_{i,j=1}^{n} C(t_i,t_j)\lambda_i\lambda_j \geqslant 0,$$

there is a Gaussian process with expectation function μ and covariance function C.

(4 points)

Exercise 3. Let $(X_t)_{t\in\mathbb{N}}$ be a sequence of r.v.'s on Ω and $a, b \in \mathbb{R}$ (or $\mathbb{Q}$) with $a < b$. For $\omega \in \Omega$ the integers $\tau_0(\omega) = 1$, $\sigma_{k+1}(\omega) = \inf\{t \geqslant \tau_k(\omega) : X_t(\omega) \leqslant a\}$ and $\tau_{k+1}(\omega) = \inf\{t \geqslant \sigma_k(\omega) : X_t(\omega) \geqslant b\}$, $k = 0, 1, 2, \dots$ define the upcrossing number associated with $(X_t(\omega))_{t\in\mathbb{N}}$, $\beta_{a,b}(\omega) = \sup\{k \geqslant 1 : \tau_k(\omega) < \infty\}$ (see §3.1.6). Show the following claims:

(a) The numbers $\beta_{a,b}(\omega)$ define a r.v. on Ω.

(b) $(X_t)_{t\in\mathbb{N}}$ converges a.s. if and only if for any $a < b$ in $\mathbb{R}$ (or $\mathbb{Q}$) the upcrossing numbers $\beta_{a,b}$ are finite a.s. for any $a < b$ in $\mathbb{R}$ (or $\mathbb{Q}$).

Let $\mathbb{F} = (\mathscr{F}_n)_{n\in\mathbb{N}}$ be a filtration and $\mathscr{F}_\infty := \bigvee_{n\in\mathbb{N}} \mathscr{F}_n$. Show that:

(c) For any positive r.v. Z we have $\mathbb{E}(Z|\mathscr{F}_n) \overset{n\to\infty}{\longrightarrow} \mathbb{E}(Z|\mathscr{F}_\infty)$ a.s. on the complement of the event $\bigcap_{n\in\mathbb{N}}\{\mathbb{E}(Z|\mathscr{F}_n) = \infty\}$.

(d) For any positive (super-)martingale $(X_n)_{n\in\mathbb{N}}$ and for any stopping time τ, the stopped process $X^\tau = (X_{\tau\wedge n})_{n\in\mathbb{N}}$ is a positive (super-)martingale. (4 points)

Exercise 4. (a) Let $(N_t)_{t\geqslant 0}$ be a Poisson process of intensity $\lambda > 0$ on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$, and let $M_t := N_t - \lambda t$. Show that $(M_t^2 - N_t)_{t\geqslant 0}$ and $(M_t^2 - \lambda t)_{t\geqslant 0}$ are both intergable martingales with respect to the filtration $(\sigma(N_s, s \leqslant t))_{t\geqslant 0}$.

(b) Let $X = (X_t)_{t\in\mathbb{T}}$, $\mathbb{T} \subseteq \mathbb{R}$ be an integrable $\mathbb{F}$-martingale with values in $\mathbb{R}$ and $\phi : \mathbb{R} \to \mathbb{R}$ a measurable convex function. Show the following claims:

(i) If $\phi(X_t)$ is integrable for all $t \in \mathbb{T}$, then $(\phi(X_t))_{t\in\mathbb{T}}$ is an $\mathbb{F}$-submartingale.

(ii) If $t^* = \sup(\mathbb{T}) \in \mathbb{T}$, and $\phi(X_{t^*})$ is integrable, then $(\phi(X_t))_{t\in\mathbb{T}}$ is an $\mathbb{F}$-submartingale.

(iii) The above statements continue to hold if X is only an integrable submartingale, but in addition ϕ is assumed to be monotone increasing. (4 points)