Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

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Exercise sheet 2

Exercise 1. (a) Let $(X_t)_{t\in\mathbb{R}^+}$, $(Y_t)_{t\in\mathbb{R}^+}$ be real-valued stochastic processes on some probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with a.s. right-continuous sample paths. Show that they are indistinguishable if and only if they are versions of each other.
Hint: For the nontrivial part, define a set $N_t = \{\omega : X_t(\omega) \neq Y_t(\omega)\}$ *for any* $t \geqslant 0$, *and consider the set* $N = \bigcup_{t\in\mathbb{Q}^+} N_t$.

(b) Find an example of two processes that are not indistinguishable, but versions of each other. (4 points)

Exercise 2. (a) Let $(\Omega, \mathscr{A}, \mathbb{P}, \mathscr{F}_{\mathbb{T}})$ be a filtered probability space, $\mathbb{T}$ countable and τ a r.v. with values in $\mathbb{T} \cup \{\infty\}$. Show that τ is a stopping time if and only if $\{\tau = t\} \in \mathscr{F}_t$ for all $t \in \mathbb{T}$.

(b) Let τ and σ be stopping times. Prove the following claims:

(i) $\tau \wedge \sigma = \min(\sigma, \tau)$ and $\tau \vee \sigma = \max(\sigma, \tau)$ are stopping times.

(ii) If $\tau, \sigma \geqslant 0$, then $\tau + \sigma$ is a stopping time.

(iii) If $s \in \mathbb{R}^+$, then $\tau + s$ is a stopping time.

(iv) For $s \in \mathbb{R}^+$, $\tau - s$ is generally not a stopping time. (4 points)

Exercise 3. Let $(\mathcal{S}, d)$ be a Polish space equipped with its Borel-σ-algebra $\mathscr{B}(\mathcal{S}) = \sigma(\mathcal{O} \subseteq \mathcal{S} : \mathcal{O} \text{ open})$ and let $\mathbb{P}$ be a probability measure on $(S, \mathscr{B}(S))$. Prove that the set

$$\mathscr{D} := \big\{B \in \mathscr{B}(\mathcal{S}) \ : \ \forall \varepsilon > 0 \, \exists \, K \subseteq B \text{ compact, } B \subseteq \mathcal{O} \text{ open, such that } \mathbb{P}(\mathcal{O} \setminus K) \leqslant \varepsilon \big\}$$

is a σ-algebra over $\mathcal{S}$. (4 points)

Exercise 4. (a) Let $\mathcal{P} := \{\mathfrak{N}(\mu, \sigma_n^2); n \in \mathbb{N}\}$ be a family of normal distributions on $(\mathbb{R}, \mathscr{B})$ with common mean $\mu \in \mathbb{R}$ and individual variances $\sigma_n^2 > 0$. Show that $\mathcal{P}$ is uniformly tight if and only if $(\sigma_n^2)_{n\in\mathbb{N}}$ is bounded.

(b) Let $(\mathcal{S}, d)$ be a metric space and $A \subseteq \mathcal{S}$. Show that the family of Dirac measures $\mathcal{Q}_A := \{\delta_a : a \in A\}$ is uniformly tight if and only if A is relatively compact. (4 points)

Handing in during the lecture on Friday, 13 May 2016