Lecture course *Probability Theory II*
Summer semester 2016
Ruprecht-Karls-Universität Heidelberg

Prof. Dr. Jan JOHANNES

Exercise sheet 1

Exercise 1. (a) Let $X : (\Omega, \mathscr{A}) \to (\mathcal{S}, \mathscr{S})$ and $Y : (\Omega, \mathscr{A}) \to (\mathbb{R}, \mathscr{B})$ be r.v.'s. Show that the real r.v. Y is $\sigma(X)$-$\mathscr{B}$-measurable if and only if there exists a $\mathscr{S}$-$\mathscr{B}$-measurable map $f : \mathcal{S} \to \mathbb{R}$ such that $Y = f(X)$.

(b) Let $X \in L_1(\Omega, \mathscr{A}, \mathbb{P})$. Prove that the family $\{\mathbb{E}(X|\mathscr{F}) : \mathscr{F} \text{ is sub-}\sigma\text{-algebra of } \mathscr{A}\}$ of r.v.'s in $L_1(\Omega, \mathscr{A}, \mathbb{P})$ is uniformly integrable.

(4 points)

Exercise 2. (a) Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space such that $\Omega = \biguplus_{i=1}^{\infty} A_i$ for countably many mutually disjoints sets $A_1, A_2, \ldots \in \mathscr{A}$. Show that for any $X \in L_1(\Omega, \mathscr{A}, \mathbb{P})$, the conditional expectation of X given $\mathscr{F} := \sigma\big(\{A_i : i \in \mathbb{N}\}\big)$ can be written as

$$\mathbb{E}(X|\mathscr{F}) = \sum_{i \in \mathbb{N}: \mathbb{P}(A_i) > 0} \frac{\mathbb{E}\big(\mathbb{1}_{A_i} X\big)}{\mathbb{P}(A_i)} \mathbb{1}_{A_i}.$$

(b) A fair die is rolled two times. Assume that a person A knows the result of this random experiment, while a person B does not. Person A answers the following questions:

- Is the product of the outcome $\geqslant 16$?
- Is the first outcome a prime number?

Person B has to estimate the *sum* of the two outcomes from the information given by A. Formulate a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ that describes this situation and a sub-σ-algebra $\mathscr{F}$ that encodes the information given by person A. Using (a), infer the conditional expectation $\mathbb{E}(X|\mathscr{F})$, where X is the sum of the two outcomes.

(4 points)

Exercise 3. (a) Let $(L_t)_{t \geqslant 0}$ and $(N_t)_{t \geqslant 0}$ denote two independent Poisson processes with intensities λ and μ, respectively. Define the process $(K_t)_{t \geqslant 0}$ by $K_t(\omega) = L_t(\omega) + N_t(\omega)$. Prove that $(K_t)_{t \geqslant 0}$ is again a Poisson process with intensity $\lambda + \mu$.

(b) Assume that the events counted by a Poisson process N_t with intensity λ can be of two distinct types (*'type I'* and *'type II'*) with probabilities p and $1 - p$. Let N_t^I and N_t^{II}, respectively, denote the number of *type I* or *type II* events in $[0, t]$. Show that both are Poisson processes, and find their intensities.

Hint: Theorem \$2.1.3 from the lecture can be useful.

(4 points)

Exercise 4. (Elementary properties of time-homogeneous Markov processes) Let X be a time-homogeneous Markov process with values in a finite set $\mathcal{S}$, and $(P(t))_{t \geqslant 0}$ its family of transition matrices. We require the transition probabilities $p_{ij}(\cdot) : [0, \infty) \to [0, 1]$ to be of class C^1 for all i, j. Show the following claims:

(a) $p'_{ij}(0) \geqslant 0$ if $i \neq j$, $p'_{ii}(0) \leqslant 0$ and $\sum_j p'_{ij}(0) = 1$ for all i.

(b) The matrix $G = (p'_{ij}(0))_{i,j}$ satisfies the following equations

$$P'(t) = P(t)G, \qquad P'(t) = GP(t) \tag{1}$$

for all $t \geqslant 0$. G is called the *(infinitesimal) generator* of the Markov process.

(c) The generator G determines $P(t)$ for all times $t \geqslant 0$, and it holds $P(t) = \exp(Gt) = \sum_{k=0}^{\infty} (tG)^k / k!$.

(4 points)