Lecture course *Statistics II*
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Exercise sheet 9

Exercise 1. Given $(X_1, \dots, X_n) \sim \mathbb{P}^{\otimes n}$ and for each $\theta \in \Theta$ a function $m_\theta : \mathcal{X} \to \mathbb{R}$ belonging to $L^1_{\mathbb{P}}$ consider $M_n(\theta) = \overline{\mathbb{P}}_n m_\theta = \frac{1}{n}\sum_{i=1}^n m_\theta(X_i)$ and $M(\theta) = \mathbb{P} m_\theta$. Show that any estimator $\widehat{\theta}_n$ such that $M_n(\widehat{\theta}_n) \leqslant M_n(\theta_o) + o_{\mathbb{P}}(1)$ converges in probability to θ_o, if the following conditions are satisfied

(a) Θ is compact,

(b) $M(\theta_o) < M(\theta)$ for all $\theta \neq \theta_o$,

(c) $\theta \to m_\theta(x)$ is continuous for all x,

(d) $\sup\{|m_\theta(X)| : \theta \in \Theta\}$ belongs to $L^1_{\mathbb{P}}$.

Exercise 2. Consider a real-valued r.v. X and a r.v. $Z \sim \mathfrak{U}([0,1])$ obeying a nonlinear regression model $\mathbb{E}_{f_\theta}(X|Z) = f_\theta(Z)$, where the regression function f_θ belongs to a parametric subset $\{f_\theta, \theta \in \Theta\}$ of $L^2([0,1])$. We assume further that $\varepsilon = X - f_\theta(Z)$ has a finite second moment and it is independent of Z. Given for each $n \in \mathbb{N}$ i.i.d. copies (X_i, Z_i), $i \in [\![1, n]\!]$, of (X, Z) denote by $\widehat{\theta}_n = \arg\min_{\theta \in \Theta} \sum_{i=1}^n (X_i - f_\theta(Z_i))^2$ the least squares estimator (LSE), if it exists. Determine sufficient conditions on the parametric family $\{f_\theta, \theta \in \Theta\}$ to ensure the consistency of the LSE.

Exercise 3. For $n \in \mathbb{N}$ consider the uniform distributions $\mathbb{P}_n = \mathfrak{U}([0,1])$ and $\mathbb{Q}_n = \mathfrak{U}([0, 1 + \frac{1}{n}])$ on the common measurable space $(\mathbb{R}, \mathscr{B})$. Show that $\mathbb{P}_n \triangleleft \triangleright \mathbb{Q}_n$ holds. Is this also true for the product probability measures $\mathbb{P}_n = \mathfrak{U}^{\otimes n}([0,1])$ and $\mathbb{Q}_n = \mathfrak{U}^{\otimes n}([0, 1 + \frac{1}{n}])$ on the common measurable space $(\mathbb{R}^n, \mathscr{B}^{\otimes n})$?

Exercise 4. Considering the product experiment $(\mathbb{R}^n, \mathscr{B}^{\otimes n}, \{\mathfrak{E}\mathrm{xp}^{\otimes n}(\theta), \theta > 0\})$ test the null hypothesis $H_0 : \theta = \theta_o$ against the alternative $H_1 : \theta > \theta_o$ using the test $\phi_n = \mathbb{1}_{\{Z_n > c_n\}}$ with $Z_n := -\frac{1}{n}\sum_{i=1}^n X_i^2$.

(i) Show that the sequence of product experiments is LAN for each $\theta_o > 0$ and derive its central sequence.

(ii) Determine a sequence c_n such that ϕ_n is an asymptotic level α test.

(iii) Show that the asymptotic power of ϕ_n under a local alternative $\theta_o + h/\sqrt{n}$ is given by $\lim_{n\to\infty} \beta_{\varphi_n}(\theta_o + h/\sqrt{n}) = \mathbb{F}_{\mathfrak{N}(0,1)}(-z_{1-\alpha} + h\frac{2}{\theta_o\sqrt{5}})$.

(iv) Calculate the relative asymptotic efficiency of ϕ_n w.r.t. an asymptotic optimal test ϕ_n^*. Which sample size n is needed, such that ϕ_n has the same asymptotic power than $\phi_{n'}^*$ with sample size $n' = 100$.

Hint: If $X \sim \mathfrak{E}xp(\theta)$, then $\mathbb{E}(X^k) = \frac{k!}{\lambda^k}$ for each $k \in \mathbb{N}$.

Exercise 5. Given an i.i.d. sample $X_1, \dots, X_n \sim \mathbb{p}$ consider a kernel density estimator $\widehat{\mathbb{p}}_h = \frac{1}{hn}\sum_{i=1}^n K(\frac{X_i - x}{h})$ of $\mathbb{p}(x)$ at a given point x, where K is a kernel and h is bandwidth. Assuming that $\mathbb{p}$ belongs to the Hölder class $\mathcal{H}(\beta, L)$ and that K is a kernel of order $l = \lfloor \beta \rfloor$ with $\lambda(|\mathrm{id}|^\beta |K|) < \infty$, show that for each $x \in \mathbb{R}$, $h > 0$ and $n \in \mathbb{N}$ it holds $|\mathrm{bias}_{\mathbb{p}}(x)| \leqslant h^\beta \frac{L}{l!}\lambda(|\mathrm{id}|^\beta |K|)$.

Exercise 6. Consider r.v.'s Y and Z obeying $Y = \sigma(Z)\varepsilon$, where ε is centred with variance 1 and independent of $Z \sim \mathfrak{U}([0,1])$. Let (X_i, Z_i), $i \in [\![1, n]\!]$, be an i.i.d. sample of (X, Z). Given a kernel K and a bandwidth h consider at a point $z \in [0, 1]$ the kernel estimator $\widehat{\sigma}^2_h(z) = \frac{1}{nh}\sum_{i=1}^n Y_i^2 K\big(\frac{Z_i - z}{h}\big)$ of the variance function $\sigma^2(z)$.

(a) Derive an upper bound for $\mathrm{bias}(z) = \mathbb{E}(\widehat{\sigma}^2_h(z)) - \sigma^2(z)$ if σ^2 belongs to an Hölder class and K is a kernel of appropriate order.

(b) Derive an upper bound for $\mathbb{V}\mathrm{ar}(\widehat{\sigma}^2_h(z))$ assuming that $\mathbb{E}(\varepsilon^4) < \infty$, $\|\sigma^2\|_{L^\infty} < \infty$ and $\|K\|_{L^2} < \infty$.

(c) Find an upper bound for $\mathrm{MSE}(z) = \mathbb{E}|\widehat{\sigma}^2_h(z) - \sigma^2(z)|^2$ depending on the bandwidth. Select an optimal value for the bandwidth and derive the associated upper bound.

Exercise 7. Consider an ONB $\{\mathbb{1}_{[0,1]}\} \cup \{u_j, j \in \mathbb{N}\}$ in $L^2[0,1]$ and the sieve $([\![1, m]\!])_{m\in\mathbb{N}}$ in $\mathbb{N}$. Given for each $n \in \mathbb{N}$ an observable quantity $[\widehat{\mathbb{p}}] = (\overline{\mathbb{P}}_n u_j)_{j\in\mathbb{N}}$ using an i.i.d. sample $X_i \sim \mathbb{p}$, $i \in [\![1, n]\!]$, let $\big\{\widehat{\mathbb{p}}_m = \mathbb{1}_{[0,1]} + \sum_{j=1}^m [\widehat{\mathbb{p}}]_j u_j, m \in \mathbb{N}\big\}$ be a family of OSE's of $\mathbb{p} = \mathbb{1}_{[0,1]} + \sum_{j\in\mathbb{N}}[\mathbb{p}]_j u_j \in L^2([0,1])$. Assuming that $0 < \mathbb{p}_0^{-1} \leqslant \mathbb{p} \leqslant \mathbb{p}_0 < \infty$ λ-a.s. for some finite constant $\mathbb{p}_0 \geqslant 1$ show that the OSE $(\widehat{\mathbb{p}}_{m_n})_{n\in\mathbb{N}}$ with $m_n := \arg\min\{\max(n^{-1}m, \sum_{j>m}|[\mathbb{p}]_j|^2), m \in \mathbb{N}\}$ is oracle optimal (up to a constant) for the integrated means squared error, i.e., $\mathbb{E}\,\|\widehat{\mathbb{p}} - \mathbb{p}\|^2_{L^2}$.

Exercise 8. Consider r.v.'s Y and Z obeying $Y = \sigma(Z)\varepsilon$, where ε is centred with variance 1 and independent of $Z \sim \mathfrak{U}([0,1])$. Let (X_i, Z_i), $i \in [\![1, n]\!]$, be an i.i.d. sample of (X, Z). Given an ONB $\{u_j, j \in \mathbb{N}\}$ in $L^2[0,1]$, the sieve $([\![1, m]\!])_{m\in\mathbb{N}}$ in $\mathbb{N}$, and for $j \in \mathbb{N}$ the observable quantity $[\widehat{\sigma}^2]_j = \frac{1}{n}\sum_{i=1}^n Y_i^2 u_j(Z_i)$ let $\big\{\widehat{\sigma}^2_m := \sum_{j=1}^m [\widehat{\sigma}^2]_j u_j, m \in \mathbb{N}\big\}$ be a family of OSE's of $\sigma^2 = \sum_{j\in\mathbb{N}}[\sigma^2]_j u_j \in L^2([0,1])$. Assuming for some finite constant $\sigma_o^2 \geqslant 1$ that $0 < \sigma_o^{-2} \leqslant \sigma^2 \leqslant \sigma_o^2 < \infty$ λ-a.s. show that the OSE $(\widehat{\sigma}^2_{m_n})_{n\in\mathbb{N}}$ with $m_n := \arg\min\{\max(n^{-1}m, \sum_{j>m}|[\sigma^2]_j|^2), m \in \mathbb{N}\}$ is oracle optimal (up to a constant) for the integrated means squared error, i.e., $\mathbb{E}\,\|\widehat{\sigma}^2 - \sigma^2\|^2_{L^2}$.