Lecture course *Statistics II*
Winter semester 2016/17
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Exercise sheet 6

Exercise 1. Given $\Theta \subset \mathbb{R}^k$ let $\mathbb{P}_\Theta$ be an exponential family on $(\Omega, \mathscr{A})$ with natural parametrisation and statistic $T : \Omega \to \mathbb{R}^k$ where for each $\theta \in \Theta$, $\mathbb{P}_\theta$ admits a likelihood function $L_\theta(x) = \exp\big(\langle\theta, T(x)\rangle - H(\theta)\big) \cdot h(x)$, $x \in \Omega$, w.r.t. a σ-finite measure μ and $\Theta = \{\theta \in \mathbb{R}^k : \mu(\exp(\langle\theta, T\rangle h) < \infty\}$ is the natural parameter space.

(i) Show that the sequence of product experiments $(\Omega^n, \mathscr{A}^{\otimes n}, \mathbb{P}_\Theta^{\otimes n})$ is LAN for each inner point θ_o of Θ and derive its central sequence.

(ii) Given $\mu_0 \in \mathbb{R}$ let $\{\mathbb{P}_b, b > 0\}$ be a family of Laplace distributions on $(\mathbb{R}, \mathscr{B})$ centred at μ_o where $\mathbb{P}_b$ has a Lebesgue-likelihood $L_b(x) = \frac{1}{2b}\exp\big(-|x-\mu_0|/b\big)$, $x \in \mathbb{R}$, for each $b > 0$. Show with (i) that the sequence of product experiments $(\Omega^n, \mathscr{A}^{\otimes n}, \{\mathbb{P}_b^{\otimes n}, b > 0\})$ is LAN in every $b_o > 0$ and determine the central sequence.

Hint: Given an exponential family with natural parametrisation the map $\theta \mapsto H(\theta)$ is infinite many times differentiable at each inner point of Θ. Moreover, it holds $\dot{H}(\theta) = \mathbb{P}_\theta T$ and $\ddot{H}(\theta) = \mathbb{P}_\theta(T - \mathbb{P}_\theta T)(T - \mathbb{P}_\theta T)^t$.

(4 points)

Exercise 2. Considering the product experiment $(\mathbb{R}^n, \mathscr{B}^{\otimes n}, \{\mathfrak{U}^{\otimes n}[0,\theta], \theta > 0\})$ test the null hypothesis $H_0 : \theta = \theta_o$ against the alternative $H_1 : \theta < \theta_o$ using the (uniformly most powerful) test $\phi_n = \mathbb{1}_{\{Z_n > c_n\}}$ with $Z_n := n(\theta_o - X_{(n)})$ and $X_{(n)} = \max\{X_i, i \in [\![1, n]\!]\}$.

(i) Show that $Z_n \xrightarrow{d} Z \sim \mathfrak{Exp}(1/\theta_o)$ under $\mathfrak{U}^{\otimes n}[0, \theta_o]$.
Hint: The c.d.f. of $X_{(n)}$ satisfies $\mathbb{F}_{X_{(n)}}(z) = \big(z/\theta_o\big)^n$ for $z \in [0, \theta_o]$.

(ii) For each $h > 0$ show that $\mathrm{d}\mathfrak{U}^{\otimes n}[0, \theta_o - h/n]/\mathrm{d}\mathfrak{U}^{\otimes n}[0, \theta_o] \xrightarrow{d} \exp\big(h/\theta_o\big)\mathbb{1}_{\{Z \geqslant h\}}$ under $\mathfrak{U}^{\otimes n}[0, \theta_o]$ and conclude that $\mathfrak{U}^{\otimes n}[0, \theta_o - h/n] \lhd \mathfrak{U}^{\otimes n}[0, \theta_o]$.

(iii) Exploit the abstract version of Le Cam's third Lemma (Theorem §3.2.5) to show that $Z_n \xrightarrow{d} L$ under $\mathfrak{U}^{\otimes n}[0, \theta_o - h/n]$ and determine the distribution of L.

(iv) Assume that $c_n \to c$ (it is not necessary to determine the sequence of critical values). Calculate the limit of the power $\beta_{\phi_n}\big(\theta_o - h/n\big)$.
Hint: If a sequence of c.d.f.'s converges point-wise, i.e., $\mathbb{F}_n(x) \xrightarrow{n\to\infty} \mathbb{F}(x)$ for all $x \in \mathbb{R}$, then it converges also uniformly, i.e., $\sup_{x\in\mathbb{R}}|\mathbb{F}_n(x) - \mathbb{F}(x)| \xrightarrow{n\to\infty} 0$.

(4 points)

Exercise 3. Consider a family of Pareto-distributions $\{\mathbb{P}_\beta, \beta > 2\}$ with likelihood $L_\beta(x) = \beta x^{-\beta-1}\mathbb{1}_{\{x>1\}}$ w.r.t. the Lebesgue measure on $\mathbb{R}$. Given $(X_1, \dots, X_n) \odot \mathbb{P}_\beta^{\otimes n}$ test the null hypothesis $H_0 : \beta = 3$ against the alternative $H_1 : \beta > 3$ using the test $\phi_n = \mathbb{1}_{\{-\overline{X}_n > d_n\}}$

with $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$. Determine a sequence d_n such that ϕ_n is an asymptotic level α test. Calculate the relative asymptotic efficiency of ϕ_n w.r.t. an asymptotic optimal test ϕ_n^*. Which sample size n is needed, such that ϕ_n has the same asymptotic power than $\phi_{n'}^*$ with sample size $n' = 100$.

Hint: You may use results obtained in Exercise 2. (4 points)

Exercise 4. Let $X_1, \ldots, X_n$ be i.i.d. r.v.'s with common density f w.r.t. the Lebesgue measure on $\mathbb{R}$ and let $R = (R_1, \ldots, R_n)$ be the associated rank vector.

(i) Show that the rank vector R and the ordered vector $(X_{R_1}, \ldots, X_{R_n})$ are independent.

(ii) Prove that the ordered vector $(X_{R_1}, \ldots, X_{R_n})$ admits a density w.r.t. the Lebesgue measure given by $n!\mathbb{1}_B(x)\prod_{i=1}^n f(x_i)$ with $B := \{(x_1, \ldots, x_n) \in \mathbb{R}, x_1 < \ldots < x_n\}$.

(4 points)

Handing in during the lecture on **Wednesday, December 14, 2016** in **fixed groups of two**.