Lecture course *Statistics II*
Winter semester 2016/17
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Exercise sheet 5

Exercise 1. Consider two sequences $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ of r.v.'s in $\mathbb{R}$. Suppose that $X_n = Y_n + o_{\mathbb{P}}(1)$ and that $X_n \xrightarrow{d} Q$ for a limiting law Q with continuous distribution function F, i.e., $\mathbb{P}(X_n \leqslant x) \overset{n\to\infty}{\longrightarrow} F(x)$ for all $x \in \mathbb{R}$. For $0 < \alpha < 1$ choose a sequence $(k_{\alpha,n})_{n\in\mathbb{N}}$ such that $\mathbb{P}(X_n \leqslant k_{\alpha,n}) = \alpha + o(1)$. Show that $\mathbb{P}(Y_n \leqslant k_{\alpha,n}) = \alpha + o(1)$.
(4 points)

Exercise 2. Consider a sequence $(Y_i)_{i\in\mathbb{N}}$ of r.v.'s obeying the following regression model

$$Y_i = \theta x_i + \varepsilon_i, \quad i \in \mathbb{N},$$

with unknown parameter $\theta \in \mathbb{R}$, deterministic explanatory variables $(x_i)_{i\in\mathbb{N}} \subseteq \mathbb{R}$ and independent and identically $\mathfrak{N}(0,1)$-distributed error terms $(\varepsilon_i)_{i\in\mathbb{N}}$. For each $n \in \mathbb{N}$ and $\theta \in \mathbb{R}$ denote by $\mathbb{P}^n_\theta$ the joint distribution of $Y_1, ..., Y_n$ defined on $(\mathbb{R}^n, \mathscr{B}^{\otimes n})$. Assume that $\sum_{i=1}^n x_i^2 \to \infty$ as $n \to \infty$. Show that the experiment $(\mathbb{R}^n, \mathscr{B}^{\otimes n}, \{\mathbb{P}^n_\theta, \theta \in \mathbb{R}\})$ is LAN in θ.
(4 points)

Exercise 3. Let $((Y_i, X_i))_{i\in\mathbb{N}}$ be an i.i.d. sequence of r.v.'s obeying the regression model

$$Y_i = \theta^1_o + \theta^2_o X_i + \varepsilon_i, \quad i \in \mathbb{N},$$

where the sequence $(X_i)_{i\in\mathbb{N}}$ of independent $\mathfrak{U}[-1,1]$-distributed r.v.'s and the sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ of independent $\mathfrak{N}(0,\sigma^2)$-distributed r.v.'s are independent. Let $\theta_o = (\theta^1_o, \theta^2_o)$ be an unknown parameter belonging to the interior of a compact set $\Theta \subseteq \mathbb{R}^2$. Denoting $q((y,x),\theta) = (y - \theta^1 - \theta^2 x)^2$ consider an M-estimator of θ_o given by

$$\hat{\theta}_n := \arg\min_{\theta\in\Theta} \frac{1}{n}\sum_{i=1}^n q((Y_i, X_i), \theta).$$

In other words $\hat{\theta}_n$ is a least square estimator of θ_o.

(1) Show that $\hat{\theta}_n$ is a consistent estimator of θ_o.
(2) Prove that $\sqrt{n}(\hat{\theta}_n - \theta_o)$ is asymptotically normal.
(3) Construct the Wald-Test for the hypothesis $A(\theta_o) = \theta^2_o = 0$ including the proof of the asymptotic level of the test.
(4 points)

Exercise 4. Let $(X_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of $\mathfrak{N}(\mu_o, \sigma^2_o)$-distributed rv's with unknown parameter $\theta_o = (\mu_o, \sigma_o)$. Find the maximum likelihood estimator for θ_o and construct the

Wald-Test based on the MLE for the hypothesis $A((\mu, \sigma)) = \mu - \sigma = 0$. Show that the asymptotic power function of this test under the distribution $\theta_o + n^{-\frac{1}{2}}h$ for $h = (1, 1)^T$ is the same as the asymptotic power on θ_o, i.e., compute and compare the limits of $\beta_n(h)$ and $\beta_n(0)$. Are you surprised by the result?

Hint: You may use results obtained in Exercise 4 on sheet 4. (4 points)

Handing in on **Friday, December 2, 2016** in **fixed groups of two**.