Lecture course *Statistics II*
Winter semester 2016/17
Ruprecht-Karls-Universität Heidelberg

Prof. Dr. Jan JOHANNES
Xavier LOIZEAU

Exercise sheet 4

Exercise 1. (a) Let $R : \mathbb{R} \to \mathbb{R}$ be continuous at zero with $R(0) = 0$. For each $n \in \mathbb{N}$ consider r.v.'s $X_{1,n}, \dots, X_{n,n}$ satisfying $\max\{|X_{i,n}|, i \in [\![1, n]\!]\} = o_{\mathbb{P}}(1)$ as $n \to \infty$. Show that $\max\{|R(X_{i,n})|, i \in [\![1, n]\!]\} = o_{\mathbb{P}}(1)$ as $n \to \infty$.

(b) Consider functions $f, f_1, f_2, \dots$ in L^2_μ satisfying (i) $\limsup_{n\to\infty} \mu f_n^2 \leqslant \mu f^2$ and (ii) $f_n(x) \to f(x)$ as $n \to \infty$ for all x. Show that, $f_n \to f$ in L^2_μ as $n \to \infty$.
Hint: Fatou's lemma

(4 points)

Exercise 2. Consider a location model $(\mathbb{R}, \mathbb{B}, \mathcal{P}_{\mathbb{R}})$ with likelihood w.r.t.. the Lebegues measure given by $L_\theta(x) = \frac{1}{2}\exp(-|x - \theta|)$, $x, \theta \in \mathbb{R}$. Show that $\mathcal{P}_{\mathbb{R}}$ is Hellinger-differentiable.
Hint: Employ $\sqrt{L_{\theta+h}(x)} - \sqrt{L_\theta(x)} = \int_0^1 \frac{1}{2} h \,\mathrm{sign}(x - \theta - uh)\sqrt{L_0(x - \theta - uh)}du$ *and proceed as in the proof of Proposition §4.2.2.*

(4 points)

Exercise 3. Consider $(\mathbb{R}, \mathbb{B}, \mathcal{P}_{\mathbb{R}})$ with likelihood w.r.t.. the Lebegues measure given by $L_\theta(x) = \theta^{-1}\mathbb{1}_{[0,\theta]}(x)$, $x, \theta \in \mathbb{R}$, i.e. $X \sim \mathbb{P}_\theta$ is uniformly distributed on the interval $[0, \theta]$. Show that $\mathcal{P}_{\mathbb{R}}$ is not Hellinger-differentiable.

(4 points)

Exercise 4. Consider a statistical scale model $(\mathbb{R}, \mathbb{B}, \mathcal{P}_{(0,\infty)})$ with likelihood w.r.t.. the Lebegues measure for each $\theta \in (0, \infty)$ given by $L_\theta(x) = \theta^{-1}g(x/\theta)$, $x \in \mathbb{R}$, where g is strictly positive.

(i) Find conditions on g such that the sequence of product experiments $(\mathbb{R}^n, \mathbb{B}^{\otimes n}, \mathcal{P}^{\otimes n}_{(0,\infty)})$ is LAN for all $\theta \in (0, \infty)$ and determine the central sequence.

(ii) Show that the sequence of Gaussian location experiments $(\mathbb{R}^n, \mathbb{B}^{\otimes n}, \{\mathfrak{N}^{\otimes n}(0, \theta^2), \theta \in (0, \infty)\})$ is LAN in every $\theta \in (0, \infty)$ and determine the central sequence.

Hint: If $X \sim \mathfrak{N}(0, 1)$ then $\mathbb{E}(X^2 - 1)^2 = 2$.

(4 points)

Handing in on **Friday, November 25, 2016** in **fixed groups of two**.