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Outline of the lecture course

PROBABILITY THEORY 1

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If you find **errors in the outline**, please send a short note
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Chapter 1

Measure and integration theory

§01 Measure theory

§01.01 **Notation.** For $x, y \in \mathbb{R}$ we agree on the following notations $\lfloor x \rfloor := \max\big\{k \in \mathbb{Z}\colon k \in (-\infty, x]\big\}$ (integer part), $x \vee y = \max(x, y)$ (maximum), $x \wedge y = \min(x, y)$ (minimum), $x^+ = \max(x, 0)$ (positive part), $x^- = \max(-x, 0)$ (negative part) and $|x| = x^- + x^+$ (modulus).

(a) For $c \in \mathbb{R}$ and $\mathbb{A} \subseteq \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} = [-\infty, \infty]$ we set $\mathbb{A}_{\geqslant c} := \mathbb{A} \cap [c, \infty]$, $\mathbb{A}_{\leqslant c} := \mathbb{A} \cap [-\infty, c]$, $\mathbb{A}_{>c} := \mathbb{A} \cap (c, \infty]$, $\mathbb{A}_{<c} := \mathbb{A} \cap [\infty, c)$, $\mathbb{A}_{\setminus c} := \mathbb{A} \setminus \{c\}$, and $\overline{\mathbb{A}} := \mathbb{A} \cup \{\pm\infty\}$.

(b) For $b \in \overline{\mathbb{R}}$ and $a \in \overline{\mathbb{R}}_{<b}$ we write $[\![a, b]\!] := [a, b] \cap \overline{\mathbb{Z}}$, $[\![a, b)\!) := [a, b) \cap \overline{\mathbb{Z}}$, $(\!(a, b]\!] := (a, b] \cap \overline{\mathbb{Z}}$, and $(\!(a, b)\!) := (a, b) \cap \overline{\mathbb{Z}}$. Moreover, let $[\![n]\!] := [\![1, n]\!]$ and $[\![n)\!) := [\![1, n)\!)$ for $n \in \mathbb{N} = \mathbb{Z}_{>0}$.

(c) $\Omega \neq \emptyset$ denotes a nonempty set, and 2^Ω the set of all subsets of Ω. A set is called *countable* if it is at most countable infinite, meaning either finite or countably infinite. The *cardinality* of a set A is denoted by $|A|$. □

§01|01 Classes of sets

§01.02 **Definition.** A class of sets $\mathscr{E} \subseteq 2^\Omega$ is called

$\cap$-closed (*closed under intersections*) or a *π-system* if $A \cap B \in \mathscr{E}$ whenever $A, B \in \mathscr{E}$,

σ-$\cap$-closed (*closed under countable intersections*) if $\cap_{n\in\mathbb{N}} A_n \in \mathscr{E}$ for any sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{E}$,

$\cup$-closed (*closed under unions*) if $A \cup B \in \mathscr{E}$ whenever $A, B \in \mathscr{E}$,

σ-$\cup$-closed (*closed under countable unions*) if $\cup_{n\in\mathbb{N}} A_n \in \mathscr{E}$ for any sequence $(A_n)_{n\in\mathbb{N}}$ of sets in $\mathscr{E}$,

$\setminus$-closed (*closed under differences*) if $A \setminus B \in \mathscr{E}$ whenever $A, B \in \mathscr{E}$, and

closed under complements if $A^{\mathrm{c}} := \Omega \setminus A \in \mathscr{E}$ for any set $A \in \mathscr{E}$. □

§01.03 **Remark.**

(a) If $\mathscr{E} \subseteq 2^\Omega$ is closed under complements then de Morgan's rule (i.e. $(\cup A_i)^{\mathrm{c}} = \cup A_i^{\mathrm{c}}$) implies immediately the equivalences of $\cup$-closed and $\cap$-closed, as well as of σ-$\cup$-closed and σ-$\cap$-closed.

(b) Let $\mathscr{E} \subseteq 2^\Omega$ be $\setminus$-closed. Then $\mathscr{E}$ is $\cap$-closed. If in addition $\mathscr{E}$ is σ-$\cup$-closed, then $\mathscr{E}$ is σ-$\cap$-closed. Any countable (respectively finite) union of sets in $\mathscr{E}$ can be expressed as a countable (respectively finite) disjoint union of sets in $\mathscr{E}$. □

§01.04 **Definition.** A class of sets $\mathscr{E} \subseteq 2^\Omega$ is called

semiring if (i) $\emptyset \in \mathscr{E}$, (ii) for any two sets $A, B \in \mathscr{E}$ the difference set $A \setminus B$ is a finite union of mutually disjoints sets in $\mathscr{E}$, and (iii) $\mathscr{E}$ is $\cap$-closed;

ring, if (R1) $\emptyset \in \mathscr{E}$, (R2) $\mathscr{E}$ is $\setminus$-closed, and (R3) $\mathscr{E}$ is $\cup$-closed;

σ-ring, if $\mathscr{E}$ is a σ-$\cup$-closed ring;

algebra, if (A1) $\Omega \in \mathscr{E}$, (A2) $\mathscr{E}$ is $\setminus$-closed, and (A3) $\mathscr{E}$ is $\cup$-closed;

σ-algebra, if $\mathscr{E}$ is a σ-$\cup$-closed algebra;

Dynkin-system or *λ-system*, if (D1) $\Omega \in \mathscr{E}$, (D2) $\mathscr{E}$ is closed under complements, and (D3) $\biguplus_{n\in\mathbb{N}} A_n \in \mathscr{E}$ for any choice of countably many pairwise disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{E}$. □

§01.05 **Remark.**

(a) Sometimes the disjoint union of sets is denoted by the symbol $\biguplus$. Note that this is not a new operation but only stresses the fact that the sets involved are mutually disjoint.

(b) For any $\Omega \neq \emptyset$ the classes $\{\emptyset, \Omega\}$ and 2^Ω are trivial examples of algebras, σ-algebras and Dynkin systems. Trivial examples of semirings, rings and σ-rings are $\{\emptyset\}$ and 2^Ω.

(c) A (set-)ring $\mathscr{R}$ equipped with the symmetric difference Δ as addition and the intersection $\cap$ as multiplication forms an Abelian algebraic ring $(\mathscr{R}, \Delta, \cap)$.

(d) A class of sets $\mathscr{A} \subseteq 2^\Omega$ is an algebra if and only if $\Omega \in \mathscr{A}$, and $\mathscr{A}$ is closed under complements and $\cap$-closed.

(e) A class of sets $\mathscr{A} \subseteq 2^\Omega$ with $\Omega \in \mathscr{A}$, which is closed under complements and σ-$\cup$-closed is a σ-algebra.

(f) Let $\mathscr{D} \subseteq 2^\Omega$ be a Dynkin-system. The condition (D2), i.e. $\mathscr{D}$ is closed under complements, can be equivalently replaced by the apparently stronger condition (D2') $B \setminus A \in \mathscr{D}$ for any $A, B \in \mathscr{D}$ with $A \subseteq B$, since each Dynkin-system satisfies also (D2'). Indeed for $A, B \in \mathscr{D}$ with $A \subseteq B$ the sets A and B^c are mutually disjoint and $B \setminus A = (A \biguplus B^c)^c \in \mathscr{D}$.

(g) Every σ-algebra also is a Dynkin-system. The converse does not apply because (D3) is required only for mutually disjoint sets. For example let $\Omega = \{1, 2, 3, 4\}$ and $\mathscr{D} = \{\emptyset, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\}, \Omega\}$. Then $\mathscr{D}$ is a Dynkin-system but is not an algebra. □

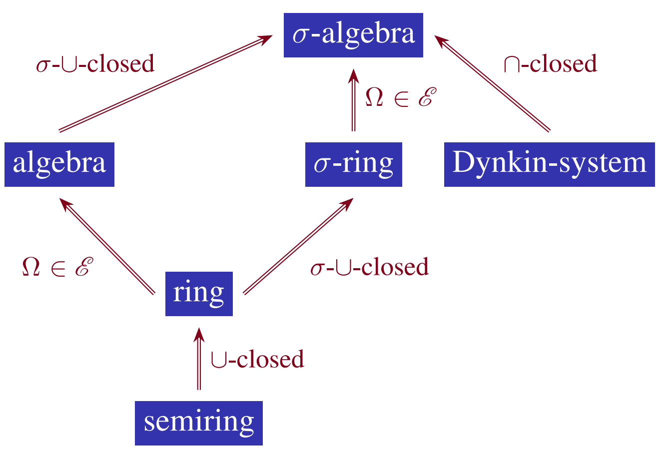
§01.6 **Illustration.**

(i) Every σ-algebra also is a Dynkin-system, an algebra and a σ-ring.

(ii) Every σ-ring is a ring, and every ring is a semiring.

(iii) Every algebra is a ring. An algebra on a finite set Ω is a σ-algebra.

Figure 01 [§01] Inclusions between classes of sets $\mathscr{E} \subseteq 2^\Omega$.



The Figure 01 [§01] was created based on Klenke (2008, Fig.1.1, p.7). □

§01.07 **Lemma**. *A Dynkin-system $\mathscr{D} \subseteq 2^\Omega$ is $\cap$-closed if and only if it is a σ-algebra.*

§01.08 **Proof** of Lemma §01.07. In the lecture course EWS. □

§01.09 **Lemma**. *Let $\mathscr{E} \subseteq 2^\Omega$ be a class of sets. Then*

$$\sigma(\mathscr{E}) := \bigcap \big\{ \mathscr{A} \colon \mathscr{A} \subseteq 2^\Omega \text{ is a } \sigma\text{-algebra and } \mathscr{E} \subseteq \mathscr{A} \big\} \quad \text{and}$$

$$\delta(\mathscr{E}) := \bigcap \big\{ \mathscr{D} \colon \mathscr{D} \subseteq 2^\Omega \text{ is a Dynkin-system and } \mathscr{E} \subseteq \mathscr{D} \big\}$$

is the smallest σ-algebra, respectively, Dynkin-system on Ω containing $\mathscr{E}$. $\mathscr{E}$ is called generator, and $\sigma(\mathscr{E})$ and $\delta(\mathscr{E})$ is called the σ-algebra and the Dynkin-system generated by $\mathscr{E}$, respectively.

§01.10 **Proof** of Lemma §01.09. In the lecture course EWS. □

§01.11 **π-λ-Theorem**. *Let $\mathscr{E} \subseteq 2^\Omega$ be $\cap$-closed. Then $\sigma(\mathscr{E}) = \delta(\mathscr{E})$ and also $\sigma(\mathscr{E}) \subseteq \mathscr{D}$ for any Dynkin-system $\mathscr{D} \subseteq 2^\Omega$ with $\mathscr{E} \subseteq \mathscr{D}$.*

§01.12 **Proof** of Theorem §01.11. In the lecture course EWS. □

§01.13 **Definition**. Let $\mathscr{E} \subseteq 2^\Omega$ be an arbitrary class of subsets of Ω and $A \in 2^\Omega \setminus \{\emptyset\} =: 2^\Omega_{\setminus\emptyset}$ a nonempty set. The class $\mathscr{E}_A := \mathscr{E}\big|_A := \mathscr{E} \cap A := \big\{ B \cap A \colon B \in \mathscr{E} \big\} \subseteq 2^\Omega$ of subsets of Ω is called *trace* of $\mathscr{E}$ on A or *restriction* of $\mathscr{E}$ to A. □

§01.14 **Remark**. If $\mathscr{E}$ is a semiring, (σ-)ring or (σ-)algebra then $\mathscr{E}_A$ is a class of sets of the same type as $\mathscr{E}$, however, on A instead of Ω. For a Dynkin-system this generally does not apply. Moreover, we have $\sigma(\mathscr{E})\big|_A = \sigma(\mathscr{E}|_A)$. □

§01.15 **Reminder**.

(a) Let $\mathcal{S}$ be a metric (or topological) space and $\mathscr{O}$ the class of open subsets in $\mathcal{S}$. The σ-algebra $\mathscr{B}_{\mathcal{S}} := \sigma(\mathscr{O})$ that is generated by the open sets $\mathscr{O}$ is called the *Borel σ-algebra* on $\mathcal{S}$. The elements of $\mathscr{B}_{\mathcal{S}}$ are called *Borel sets* or *Borel measurable sets*.

(b) In many cases, we are interested in the Borel σ-algebra $\mathscr{B}^n := \mathscr{B}_{\mathbb{R}^n}$ over $\mathbb{R}^n$, where $\mathbb{R}^n$ is equipped with the Euclidean distance $d(x, y) = \|x - y\| = \sqrt{\sum_{i \in [\![n]\!]}(x_i - y_i)^2}$ for $x = (x_i)_{i \in [\![n]\!]}, y = (y_i)_{i \in [\![n]\!]} \in \mathbb{R}^n$.

(c) For $a = (a_i)_{i \in [\![n]\!]}, b = (b_i)_{i \in [\![n]\!]} \in \overline{\mathbb{R}}^n$ we write $a < b$, if $a_i < b_i$ for all $i \in [\![n]\!]$. For $a < b$, define the open *rectangle* as the Cartesian product $(a, b) := \bigtimes_{i \in [\![n]\!]}(a_i, b_i) := (a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_n, b_n)$. Analogously, we define $[a, b]$, $(a, b]$ and $[a, b)$. Moreover, we set $(-\infty, b) := \bigtimes_{i \in [\![n]\!]}(-\infty, b_i)$ and $(-\infty, b] := \bigtimes_{i \in [\![n]\!]}(-\infty, b_i]$.

(d) The Borel σ-algebra $\mathscr{B}^n$ is generated by any of the classes of sets:
(i) $\mathscr{E}_1 := \big\{ A \subseteq \mathbb{R}^n \colon A \text{ is closed} \big\}$; (ii) $\mathscr{E}_2 := \big\{ A \subseteq \mathbb{R}^n \colon A \text{ is compact} \big\}$;
(iii) $\mathscr{E}_3 := \big\{ (a, b) \colon a, b \in \mathbb{Q}^n, a < b \big\}$; (iv) $\mathscr{E}_4 := \big\{ [a, b] \colon a, b \in \mathbb{Q}^n, a < b \big\}$;
(v) $\mathscr{E}_5 := \big\{ (a, b] \colon a, b \in \mathbb{Q}^n, a < b \big\}$; (vi) $\mathscr{E}_6 := \big\{ [a, b) \colon a, b \in \mathbb{Q}^n, a < b \big\}$;
(vii) $\mathscr{E}_7 := \big\{ (-\infty, b] \colon b \in \mathbb{Q}^n \big\}$; (viii) $\mathscr{E}_8 := \big\{ (-\infty, b) \colon b \in \mathbb{Q}^n \big\}$;
(ix) $\mathscr{E}_9 := \big\{ (a, \infty) \colon a \in \mathbb{Q}^n \big\}$ and (x) $\mathscr{E}_{10} := \big\{ [a, \infty) \colon a \in \mathbb{Q}^n \big\}$. (Exercise).

(e) We denote by $\overline{\mathscr{B}} := \mathscr{B}_{\overline{\mathbb{R}}}$ the Borel σ-algebra over the extension $\overline{\mathbb{R}} := [-\infty, \infty]$ of the real line by the points $\{\pm\infty\}$ where in $\overline{\mathbb{R}}$ the sets $\{-\infty\}$ and $\{\infty\}$ are closed, and $\mathbb{R}$ is open. In particular, $\mathscr{B} := \mathscr{B}_{\mathbb{R}} = \overline{\mathscr{B}} \cap \mathbb{R}$ is the Borel σ-algebra over $\mathbb{R}$. For $c \in \mathbb{R}$ and σ-algebra $\mathscr{A} \subseteq 2^{\overline{\mathbb{R}}}$ we write $\mathscr{A}_{\geqslant c} := \mathscr{A} \cap \overline{\mathbb{R}}_{\geqslant c}$, $\mathscr{A}_{>c} := \mathscr{A} \cap \overline{\mathbb{R}}_{>c}$, $\mathscr{A}_{\leqslant c} := \mathscr{A} \cap \overline{\mathbb{R}}_{\leqslant c}$, and $\mathscr{A}_{<c} := \mathscr{A} \cap \overline{\mathbb{R}}_{<c}$ □

§01|02 Set functions

§01.16 **Definition**. Let $\mathscr{E} \subseteq 2^\Omega$ and let $\mu : \mathscr{E} \to \overline{\mathbb{R}}_{\geqslant 0} = [0,\infty]$ be a set function. We say that μ is

monotone if $\mu(A) \leqslant \mu(B)$ for any two sets $A, B \in \mathscr{E}$ with $A \subseteq B$,

additive if $\mu(\biguplus_{j\in[\![n]\!]} A_j) = \sum_{j\in[\![n]\!]} \mu(A_j)$ for any choice of finitely many mutually disjoint sets $A_j \in \mathscr{E}$, $j \in [\![n]\!]$, with $\biguplus_{j\in[\![n]\!]} A_j \in \mathscr{E}$,

σ-additive if $\mu(\biguplus_{j\in\mathbb{N}} A_j) = \sum_{j\in\mathbb{N}} \mu(A_j)$ for any choice of countably many mutually disjoint sets $A_j \in \mathscr{E}$, $j \in \mathbb{N}$, with $\biguplus_{j\in\mathbb{N}} A_j \in \mathscr{E}$,

subadditive if $\mu(A) \leqslant \sum_{i\in[\![n]\!]} \mu(A_j)$ for any choice of finitely many sets $A, A_j \in \mathscr{E}$, $j \in [\![n]\!]$, with $A \subseteq \bigcup_{j\in[\![n]\!]} A_j$,

σ-subadditive if $\mu(A) \leqslant \sum_{j\in\mathbb{N}} \mu(A_j)$ for any choice of countably many sets $A, A_j \in \mathscr{E}$, $j \in \mathbb{N}$, with $A \subseteq \bigcup_{j\in\mathbb{N}} A_j$. □

§01.17 **Definition**. Let $\mathscr{E} \subseteq 2^\Omega$ be a semiring. A set function $\mu : \mathscr{E} \to \overline{\mathbb{R}}_{\geqslant 0}$ with $\mu(\emptyset) = 0$ is called a

content if μ is additive,

premeasure if μ is σ-additive,

measure if μ is a premeasure and $\mathscr{E}$ is a σ-algebra, and

probability measure if μ is a measure and $\mu(\Omega) = 1$.

We denote by $\mathfrak{M}(\mathscr{E})$ the set of all premeasures on $(\Omega, \mathscr{E})$. A content μ on $\mathscr{E}$ is called

finite if $\mu(A) \in \mathbb{R}_{\geqslant 0}$ for every $A \in \mathscr{E}$ and

σ-finite if there exists a sequence of sets $(\mathcal{E}_j)_{j\in\mathbb{N}}$ in $\mathscr{E}$ such that $\Omega = \bigcup_{j\in\mathbb{N}} \mathcal{E}_j$ and $\mu(\mathcal{E}_j) \in \mathbb{R}_{\geqslant 0}$ for all $j \in \mathbb{N}$.

We denote by $\mathfrak{M}_{\mathrm{f}}(\mathscr{E})$ and $\mathfrak{M}_\sigma(\mathscr{E})$ the set of all finite, respectively, σ-finite premeasures on $(\Omega, \mathscr{E})$. Moreover, for a σ-algebra $\mathscr{A} \subseteq 2^\Omega$ we denote by $\mathcal{W}(\mathscr{A})$ the set of all probability measures on $(\Omega, \mathscr{A})$.

§01.18 **Example**.

(a) For $A \in 2^\Omega$ we denote by $\mathbb{1}_A : \Omega \to \{0,1\}$ with $\mathbb{1}_A^{-1}(\{1\}) = A$ and $\mathbb{1}_A^{-1}(\{0\}) = A^{\mathrm{c}}$ the *indicator function* on A. For any σ-algebra $\mathscr{A} \subseteq 2^\Omega$ and $\omega \in \Omega$ the set function $\delta_\omega : \mathscr{A} \to \{0,1\}$ with $\delta_\omega(A) := \mathbb{1}_A(\omega)$ is a probability measure on $\mathscr{A}$. $\delta_\omega \in \mathcal{W}(\mathscr{A})$ is called the *Dirac measure* for the point ω.

(b) Let $\Omega \neq \emptyset$ be countably infinite and let $\mathscr{E} := \{A \in 2^\Omega : (|A| \wedge |A^{\mathrm{c}}|) \in \mathbb{Z}_{\geqslant 0}\}$. Then $\mathscr{E}$ is an algebra. The set function $\nu : \mathscr{E} \to \{0,\infty\}$ is given by $\nu(A) = 0$ for $A \in \mathscr{E}$ with $|A| \in \mathbb{R}_{\geqslant 0}$ and $\nu(A) = \infty$ for $|A^{\mathrm{c}}| \in \mathbb{R}_{\geqslant 0}$. Then ν is a content, but it is not a premeasure. Indeed, ν is not σ-additive, since $\nu(\Omega) = \infty$ and $\sum_{\omega\in\Omega} \nu(\{\omega\}) = 0$.

(c) Let $\Omega \neq \emptyset$ be countable and let $\mathbb{p} : \Omega \to \mathbb{R}_{\geqslant 0}$. Then $\mu : 2^\Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ with $A \mapsto \mu(A) := \sum_{\omega\in\Omega} \mathbb{p}(\omega)\delta_\omega(A)$ is a σ-finite measure on 2^Ω, i.e. $\mu \in \mathfrak{M}_\sigma(2^\Omega)$. We call $\mathbb{p}$ the *mass function* of μ. The number $\mathbb{p}(\omega)$ is called the mass of μ at point ω. Remember, if in addition $\mathbb{p}$ satisfies $\sum_{\omega\in\Omega} \mathbb{p}(\omega) = 1$ then $\mu \in \mathcal{W}(2^\Omega)$ is a discrete probability measure. If $\mathbb{p}(\omega) = 1$ for every $\omega \in \Omega$, then $\zeta_\Omega := \sum_{\omega\in\Omega} \delta_\omega$ is called *counting measure* on Ω. Evidently, if Ω is finite, then so is $\mu \in \mathfrak{M}_{\mathrm{f}}(2^\Omega)$. If $\Omega \subseteq \mathbb{R}$ then for each $\omega \in \Omega$ the dirac measure $\delta_\omega \in \mathcal{W}(\mathscr{B})$, and hence $\mu, \zeta_\Omega \in \mathfrak{M}_\sigma(\mathscr{B})$ are also called discrete measures on $(\mathbb{R}, \mathscr{B})$.

(d) For arbitrary measures $\mu, \nu \in \mathfrak{M}(\mathscr{A})$ the set function $\nu + \mu : \mathscr{A} \to \overline{\mathbb{R}}_{\geqslant 0}$ given by $(\nu + \mu)(A) = \nu(A) + \mu(A)$ for all $A \in \mathscr{A}$ is a measure. □

§01.19 **Lemma**. *Let $\mathscr{E}$ be a semiring and let μ be a content on $\mathscr{E}$. Then the following statements hold.*

(i) *If $\mathscr{E}$ is a ring, then $\mu(A \cup B) = \mu(A) + \mu(B \setminus A)$ and $\mu(B) = \mu(A \cap B) + \mu(B \setminus A)$, hence $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ for any two sets $A, B \in \mathscr{E}$.*

(ii) *μ is monotone. If $\mathscr{E}$ is a ring, then $\mu(B) = \mu(A) + \mu(B \setminus A)$ for any two sets $A, B \in \mathscr{E}$ with $A \subseteq B$.*

(iii) *μ is subadditive. If μ is σ-additive, then μ is also σ-subadditive.*

(iv) *If $\mathscr{E}$ is a ring, then $\sum_{j \in [\![n]\!]} \mu(A_j) = \mu(\biguplus_{j \in [\![n]\!]} A_j) \leqslant \mu(\biguplus_{j \in \mathbb{N}} A_j)$ for all $n \in \mathbb{N}$, and hence $\sum_{j \in \mathbb{N}} \mu(A_j) \leqslant \mu(\biguplus_{j \in \mathbb{N}} A_j)$, for any choice of countably many mutually disjoint sets $A_j \in \mathscr{E}$, $j \in \mathbb{N}$, with $\biguplus_{j \in \mathbb{N}} A_j \in \mathscr{E}$.*

(v) *If $\mathscr{E}$ is a ring, then for any $n \in \mathbb{N}$ and $(A_i)_{i \in [\![n]\!]}$ in $\mathscr{E}$ with $\mu(\bigcup_{i \in [\![n]\!]} A_i) \in \mathbb{R}_{\geqslant 0}$ the Inclusion-exclusion formulas (Poincaré and Sylvester) hold:*

$$\mu(\bigcup_{i \in [\![n]\!]} A_i) = \sum_{\mathcal{I} \in 2^{[\![n]\!]}_{\setminus \emptyset}} (-1)^{|\mathcal{I}|-1} \mu(\bigcap_{i \in \mathcal{I}} A_i) \quad \text{and} \quad \mu(\bigcap_{i \in [\![n]\!]} A_i) = \sum_{\mathcal{I} \in 2^{[\![n]\!]}_{\setminus \emptyset}} (-1)^{|\mathcal{I}|-1} \mu(\bigcup_{i \in \mathcal{I}} A_i).$$

§01.20 **Proof** of Lemma §01.19. (i), (ii) and (iv) are given in the lecture, (iii) and (v) are exercises. □

§01.21 **Notation**. We agree on the following conventions.

(a) A sequence $(x_n)_{n \in \mathbb{N}}$ in $\overline{\mathbb{R}}$ is called *increasing* (respectively *decreasing*), if $x_n \leqslant x_{n+1}$ (respectively $x_{n+1} \leqslant x_n$) for all $n \in \mathbb{N}$. If an increasing (respectively decreasing) sequence $(x_n)_{n \in \mathbb{N}}$ is convergent, say $x = \lim_{n \to \infty} x_n$, then we write $x_n \uparrow x$ (respectively $x_n \downarrow x$) for short.

(b) A sequence $(A_n)_{n \in \mathbb{N}}$ in 2^{Ω} is called *increasing* (respectively *decreasing*), if $A_n \subseteq A_{n+1}$ (respectively $A_{n+1} \subseteq A_n$) for all $n \in \mathbb{N}$. We call

$$A_\star := \liminf_{n \to \infty} A_n := \bigcup_{n \in \mathbb{N}} \bigcap_{m \in \mathbb{N}_{\geqslant n}} A_m := \bigcup \big\{ \bigcap \big\{ A_m : m \in \mathbb{N}_{\geqslant n} \big\} : n \in \mathbb{N} \big\} \quad \text{and}$$

$$A^\star := \limsup_{n \to \infty} A_n := \bigcap_{n \in \mathbb{N}} \bigcup_{m \in \mathbb{N}_{\geqslant n}} A_m$$

limes inferior, respectively, *limes superior* of the sequence $(A_n)_{n \in \mathbb{N}}$. The sequence $(A_n)_{n \in \mathbb{N}}$ is called *convergent*, if $A_\star = A^\star =: A$. In this case we write $\lim_{n \to \infty} A_n = A$ for short.

An increasing (respectively decreasing) sequence $(A_n)_{n \in \mathbb{N}}$ in 2^{Ω} is convergent with $A := \lim_{n \to \infty} A_n = \bigcup_{n \in \mathbb{N}} A_n$ (respectively $A := \lim_{n \to \infty} A_n = \bigcap_{n \in \mathbb{N}} A_n$). In this case we write $A_n \uparrow A$ (respectively $A_n \downarrow A$).

(c) For functions $f, g : \Omega \to \overline{\mathbb{R}}$ we write $f \leqslant g$ if $f(\omega) \leqslant g(\omega)$ for any $\omega \in \Omega$. Analogously, we write $f \geqslant 0$ and so on. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions on Ω is called *(pointwise) increasing*, or briefly *isotone* (respectively, *(pointwise) decreasing*, or briefly *antitone*) if $f_n \leqslant f_{n+1}$ (respectively, $f_{n+1} \leqslant f_n$) for all $n \in \mathbb{N}$. We denote by

$$f_\star := \liminf_{n \to \infty} f_n := \sup \big\{ \inf \big\{ f_m : m \in \mathbb{N}_{\geqslant n} \big\} : n \in \mathbb{N} \big\} \text{ and}$$

$$f^\star := \limsup_{n \to \infty} f_n := \sup \big\{ \inf \big\{ f_m : m \in \mathbb{N}_{\geqslant n} \big\} : n \in \mathbb{N} \big\}$$

the *limes inferior*, respectively, *limes superior*. The sequence $(f_n)_{n\in\mathbb{N}}$ is *convergent* if $f_\star = f^\star =: f$, that is, the pointwise limit exists everywhere. In this case we write $\lim_{n\to\infty} f_n = f$.

An isotone (respectively, antitone) sequence $(f_n)_{n\in\mathbb{N}}$ is convergent with $f := \lim_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} f_n$ (respectively, $f := \lim_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} f_n$). In this case we briefly write $f_n \uparrow f$ (respectively, $f_n \downarrow f$). □.

§01.22 **Definition**. A content μ on a ring $\mathscr{R} \subseteq 2^\Omega$ is called

lower semicontinuous if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ for any $A \in \mathscr{R}$ and any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{R}$ with $A_n \uparrow A$.

upper semicontinuous if $\lim_{n\to\infty} \mu(A_n) = \mu(A)$ for any $A \in \mathscr{R}$ and any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{R}$ with $\mu(A_n) \in \mathbb{R}_{\geqslant 0}$ for some (and then eventually all) $n \in \mathbb{N}$ and $A_n \downarrow A$.

$\emptyset$-continuous if $\lim_{n\to\infty} \mu(A_n) = 0 = \mu(\emptyset)$ for any sequence $(A_n)_{n\in\mathbb{N}}$ in $\mathscr{R}$ with $\mu(A_n) \in \mathbb{R}_{\geqslant 0}$ for some (and then eventually all) $n \in \mathbb{N}$ and $A_n \downarrow \emptyset$.

§01.23 **Remark**. In the definition of upper semicontinuity, we needed the assumption $\mu(A_n) \in \mathbb{R}_{\geqslant 0}$ since otherwise we would not even have $\emptyset$-continuity for an example as simple as the counting measure $\zeta_{\mathbb{N}}$ on $(\mathbb{N}, 2^{\mathbb{N}})$. Indeed, $A_n := \mathbb{N}_{\geqslant n} \downarrow \emptyset$ but $\zeta_{\mathbb{N}}(A_n) = \infty$ for all $n \in \mathbb{N}$. □

§01.24 **Lemma**. *Let μ be a content on the ring $\mathscr{R} \subseteq 2^\Omega$. Consider the following five properties.* (p1) *μ is σ-additive (and hence $\mu \in \mathfrak{M}(\mathscr{R})$ is a premeasure),* (p2) *μ is σ-subadditive,* (p3) *μ is lower semicontinuous,* (p4) *μ is $\emptyset$-continuous,* (p5) *μ is upper semicontinuous. Then the following implications hold:* (p1)⇔(p2)⇔(p3)⇒(p4)⇔(p5). *If μ is finite, then we also have* (p4)⇒(p3).

§01.25 **Proof** of Lemma §01.24. is given in the lecture. □

§01.26 **Example** *(§01.18* (b) *continued).* ν is a $\emptyset$-continuous content, but it is not a premeasure. □

§01.27 **Definition**.

(a) A pair $(\Omega, \mathscr{A})$ consisting of a nonempty set Ω and a σ-algebra $\mathscr{A} \subseteq 2^\Omega$ is called a *measurable space*. The sets $A \in \mathscr{A}$ are called *measurable sets*. If Ω is at most countably infinite and if $\mathscr{A} = 2^\Omega$, then the measurable space $(\Omega, 2^\Omega)$ is called *discrete*.

(b) A triple $(\Omega, \mathscr{A}, \mu)$ is called *measure space* if $(\Omega, \mathscr{A})$ is a measurable space and $\mu \in \mathfrak{M}(\mathscr{A})$ is a measure on $\mathscr{A}$.

(c) If in addition $\mu(\Omega) = 1$, then $(\Omega, \mathscr{A}, \mu)$ is called a *probability space* and $\mu \in \mathcal{W}(\mathscr{A})$ a *probability measure*. In this case, the sets $A \in \mathscr{A}$ are called *events*. □

§01|03 Measure extension

§01.28 **Lemma** *(Uniqueness). Let $(\Omega, \mathscr{A})$ be a measurable space, let $\mathscr{E} \subseteq \mathscr{A}$ be a $\cap$-closed generator of $\mathscr{A}$ and let $\mu, \nu \in \mathfrak{M}_\sigma(\mathscr{A})$ be two σ-finite measures on $\mathscr{A}$, which agree on $\mathscr{E}$, that is, $\mu(E) = \nu(E)$ for all $E \in \mathscr{E}$. Assume* (uC) *there exist sets $(\mathcal{E}_n)_{n\in\mathbb{N}}$ in $\mathscr{E}$ with $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n = \Omega$ and $\mu(\mathcal{E}_n) \in \mathbb{R}_{\geqslant 0}$ for all $n \in \mathbb{N}$. Then μ and ν agree also on $\mathscr{A}$.*

If $\mu, \nu \in \mathcal{W}(\mathscr{A})$ are two probability measures on $\mathscr{A}$, then (uC) *is not needed.*

§01.29 **Proof** of Lemma §01.28. is given in the lecture. □

§01.30 **Remark**. In other words under the assumptions of Lemma §01.28 a σ-finite measure $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is uniquely determined by its values $\mu(E)$, $E \in \mathscr{E}$. The uniqueness without (uC), the existence of the sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$, does generally not apply, even if $\mu \in \mathfrak{M}_f(\mathscr{A})$ is a finite measure on $\mathscr{A}$. In this case the total mass $\mu(\Omega)$ is generally not uniquely determined. Let $\Omega = \{1, 2\}$

and $\mathscr{E} = \{\{1\}\}$. Then $\mathscr{E}$ is a $\cap$-closed generator of 2^Ω. A probability measure $\mu \in \mathcal{W}(\mathscr{A})$ is uniquely determined by the value $\mu(\{1\})$. However, a finite measure is not determined by its value on $\{\{1\}\}$, as $\mu \equiv 0$ and $\nu = \delta_2$ are different finite measures that agree on $\mathscr{E}$. □

§01.31 **Definition**. A set function $\mu^\star : 2^\Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ is called an *outer measure* if (oM1) $\mu^\star(\emptyset) = 0$, (oM2) $\mu^\star$ is monotone, and (oM3) $\mu^\star$ is σ-subadditive. A set $A \in 2^\Omega$ is called *$\mu^\star$-measurable* if

$$\mu^\star(A \cap B) + \mu^\star(A^c \cap B) = \mu^\star(B) \quad \text{for any } B \in 2^\Omega.$$

We write $\sigma(\mu^\star) := \{A \in 2^\Omega : A \text{ is } \mu^\star\text{-measurable}\}$. □

§01.32 **Remark**. Since $\mu^\star(\emptyset) = 0$ we evidently have $\Omega \in \sigma(\mu^\star)$. As $\mu^\star$ is subadditive it follows that $A \in \sigma(\mu^\star)$ if and only if $\mu^\star(A \cap B) + \mu^\star(A^c \cap B) \leqslant \mu^\star(B)$ for any $B \in 2^\Omega$. □

§01.33 **Lemma**. *Let $\mathscr{E} \subseteq 2^\Omega$ be an arbitrary class of sets with $\emptyset \in \mathscr{E}$ and let $\mu : \mathscr{E} \to \overline{\mathbb{R}}_{\geqslant 0}$ be a set function with $\mu(\emptyset) = 0$. For $A \in 2^\Omega$ define the set of countable coverings $\mathscr{F}$ of A with sets $F \in \mathscr{E}$:*

$$\mathcal{U}(A) = \{\mathscr{F} \subseteq \mathscr{E} : \mathscr{F} \text{ is countable and } A \subseteq \textstyle\bigcup_{F \in \mathscr{F}} F\}$$

Define

$$\mu^\star : 2^\Omega \to \overline{\mathbb{R}}_{\geqslant 0} \text{ with } A \mapsto \mu^\star(A) := \inf\Big\{\sum_{F \in \mathscr{F}} \mu(F) : \mathscr{F} \in \mathcal{U}(A)\Big\},$$

where $\inf \emptyset = \infty$. Then $\mu^\star$ is an outer measure. If in addition μ is σ-subadditive, then $\mu^\star$ and μ agree on $\mathscr{E}$, i.e. $\mu^\star(E) = \mu(E)$ for all $E \in \mathscr{E}$.

§01.34 **Proof** of Lemma §01.33. is given in the lecture. □

§01.35 **Lemma**. *If $\mu^\star$ is an outer measure, then $\sigma(\mu^\star)$ is a σ-algebra and the restriction of $\mu^\star$ on $\sigma(\mu^\star)$ is a measure.*

§01.36 **Proof** of Lemma §01.35. is given in the lecture. □

§01.37 **Extension theorem for measures**. *Let $\mathscr{E} \subseteq 2^\Omega$ be a semiring and let $\mu : \mathscr{E} \to \overline{\mathbb{R}}_{\geqslant 0}$ be an additive, σ-subadditive and σ-finite set function with $\mu(\emptyset) = 0$.*

Then there is a unique σ-finite measure $\widetilde{\mu} : \sigma(\mathscr{E}) \to \overline{\mathbb{R}}_{\geqslant 0}$ such that $\widetilde{\mu}$ and μ agree on $\mathscr{E}$, i.e. $\widetilde{\mu}(E) = \mu(E)$ for all $E \in \mathscr{E}$.

§01.38 **Proof** of Theorem §01.37. is given in the lecture. □

§01.39 **Example**.

(a) There exists a uniquely determined measure λ^n on $(\mathbb{R}^n, \mathscr{B}^n)$ with the property that $\lambda^n((a,b]) = \prod_{i \in [\![n]\!]}(b_i - a_i)$ for all $a, b \in \mathbb{R}^n$ with $a < b$. λ^n is called *Lebesgue measure* on $(\mathbb{R}^n, \mathscr{B}^n)$ (see lecture Analysis 3).

(b) Let $\mathbb{F} : \mathbb{R} \to \mathbb{R}$ be monotone increasing and right continuous. There is a uniquely determined measure $\mu_{\mathbb{F}}$ on $(\mathbb{R}, \mathscr{B})$ with the property that $\mu_{\mathbb{F}}((a,b]) = \mathbb{F}(b) - \mathbb{F}(a)$ for all $a, b \in \mathbb{R}$ with $a < b$. $\mu_{\mathbb{F}}$ is called *Lebesgue-Stieltjes measure* on $(\mathbb{R}, \mathscr{B})$ (Exercise). If in addition $\lim_{x \to \infty}(\mathbb{F}(x) - \mathbb{F}(-x)) = 1$, then $\mu_{\mathbb{F}}$ is a probability measure. □

§01.40 **Definition**. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space.

(a) A set $N \in \mathscr{A}$ is called a *μ-null set*, or briefly null set, if $\mu(N) = 0$. By $\mathcal{N}_\mu$ we denote the class of all subsets of μ-null sets.

(b) Let $E(\omega)$ be a property that a point $\omega \in \Omega$ can have or not have. We say that E holds *μ-almost everywhere (μ-a.e.)* if there exists a μ-null set $N \in \mathcal{N}_\mu$ such that $E(\omega)$ holds for every $\omega \in \Omega \setminus N = N^c$. If $A \in \mathscr{A}$ and if there exists a μ-null set N such that $E(\omega)$ holds for every $\omega \in A \setminus N$, then we say that E holds μ-almost everywhere on A. If $\mu = \mathbb{P} \in \mathcal{W}(\mathscr{A})$ is a probability measure then we say that E holds *$\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.)* respectively $\mathbb{P}$-almost surely on A.

(c) The measure space $(\Omega, \mathscr{A}, \mu)$ is called *complete*, if $\mathcal{N}_\mu \subseteq \mathscr{A}$. □

§01.41 **Remark.** Let $(\Omega, \mathscr{A}, \mu)$ be a σ-finite measure space. There exists a unique smallest σ-algebra $\mathscr{A}^\star \supseteq \mathscr{A}$ and an extension $\mu^\star$ of μ to $\mathscr{A}^\star$ such that $(\Omega, \mathscr{A}^\star, \mu^\star)$ is complete. $(\Omega, \mathscr{A}^\star, \mu^\star)$ is called the completion of $(\Omega, \mathscr{A}, \mu)$. With the notation of Theorem §01.37, this completion is $(\Omega, \sigma(\mu^\star), \mu^\star|_{\sigma(\mu^\star)})$. Furthermore, $\sigma(\mu^\star) = \sigma(\mathscr{A} \cup \mathcal{N}_\mu) = \{A \cup N : A \in \mathscr{A}, N \in \mathcal{N}_\mu\}$ and $\mu^\star(A \cup N) = \mu(A)$ for any $A \in \mathscr{A}$ and $N \in \mathcal{N}_\mu$. □

§01.42 **Definition.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and $B \in \mathscr{A}$. On the trace σ-algebra $\mathscr{A}_B$ we define a measure by $\mu_B(A) := \mu(A)$ for $A \in \mathscr{A}$ with $A \subseteq B$. This measure is called the *restriction* of μ to B. □

§01.43 **Example.** The restriction $\lambda_{[0,1]}$ of the Lebesgue-Borel measure λ on $(\mathbb{R}, \mathscr{B})$ to $[0,1]$ is a probability measure on $([0,1], \mathscr{B}_{[0,1]})$, i.e. $\lambda_{[0,1]} \in \mathcal{W}(\mathscr{B}_{[0,1]})$. More generally, for a Borel set $B \in \mathscr{B}$ we call the restriction λ_B the Lebesgue measure on B, i.e. $\lambda_B \in \mathfrak{M}_\sigma(\mathscr{B}_B)$. □

§02 Integration theory

§02|01 The integral

§02.01 **Reminder.** Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $(\mathcal{S}, \mathscr{S})$ be a measurable space.

(a) A function $f : \Omega \to \mathcal{S}$ is called *$\mathscr{A}$-$\mathscr{S}$-measurable* (or, briefly, *measurable*) if

$$\sigma(f) := f^{-1}(\mathscr{S}) := \{f^{-1}(S) : S \in \mathscr{S}\} \subseteq \mathscr{A}.$$

If f is measurable, we write $f : (\Omega, \mathscr{A}) \to (\mathcal{S}, \mathscr{S})$. We denote by $\mathcal{M}(\mathscr{A}, \mathscr{S})$ the set of all $\mathscr{A}$-$\mathscr{S}$-measurable functions. If $\mathscr{S} = \mathscr{B}_\mathcal{S}$ is the Borel σ-algebra on $\mathcal{S}$ then we write $\mathcal{M}_\mathcal{S}(\mathscr{A}) := \mathcal{M}(\mathscr{A}, \mathscr{B}_\mathcal{S})$ for short. If $\mu = \mathbb{P} \in \mathcal{W}(\mathscr{A})$ is a probability measure then $f \in \mathcal{M}(\mathscr{A}, \mathscr{S})$ is called ($(\mathcal{S}, \mathscr{S})$-valued) *random variable*. The σ-algebra $\sigma(f)$ is called the σ-algebra on Ω that is *generated* by f. This is the smallest σ-algebra with respect to which f is measurable.

(b) The identity map $\mathrm{id}_\Omega : \Omega \to \Omega$ is $\mathscr{A}$-$\mathscr{A}$-measurable. If $\mathscr{A} = 2^\Omega$ or $\mathscr{S} = \{\emptyset, \mathcal{S}\}$, then any map $f : \Omega \to \mathcal{S}$ belongs to $\mathcal{M}(\mathscr{A}, \mathscr{S})$. The indicator function $\mathbb{1}_A$ for $A \in 2^\Omega$ belongs to $\mathcal{M}(\mathscr{A}, 2^{\{0,1\}})$ if and only if $A \in \mathscr{A}$.

(c) A measurable function $f : (\Omega, \mathscr{A}) \to (\mathcal{S}, \mathscr{S})$ is called

numerical if $(\mathcal{S}, \mathscr{S}) = (\overline{\mathbb{R}}, \overline{\mathscr{B}})$, briefly $f \in \overline{\mathcal{M}}(\mathscr{A}) := \mathcal{M}_{\overline{\mathbb{R}}}(\mathscr{A}) = \mathcal{M}(\mathscr{A}, \overline{\mathscr{B}})$,

positive numerical if $(\mathcal{S}, \mathscr{S}) = (\overline{\mathbb{R}}_{\geqslant 0}, \overline{\mathscr{B}}_{\geqslant 0})$, briefly $f \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) := \mathcal{M}_{\overline{\mathbb{R}}_{\geqslant 0}}(\mathscr{A}) = \mathcal{M}(\mathscr{A}, \overline{\mathscr{B}}_{\geqslant 0})$,

real if $(\mathcal{S}, \mathscr{S}) = (\mathbb{R}, \mathscr{B})$, briefly $f \in \mathcal{M}(\mathscr{A}) := \mathcal{M}_\mathbb{R}(\mathscr{A}) = \mathcal{M}(\mathscr{A}, \mathscr{B})$,

positive real if $(\mathcal{S}, \mathscr{S}) = (\mathbb{R}_{\geqslant 0}, \mathscr{B}_{\geqslant 0})$, briefly $f \in \mathcal{M}_{\geqslant 0}(\mathscr{A}) := \mathcal{M}_{\mathbb{R}_{\geqslant 0}}(\mathscr{A}) = \mathcal{M}(\mathscr{A}, \mathscr{B}_{\geqslant 0})$.

If the preimage $(\Omega, \mathscr{A})$ is irrelevant we also write shortly $\overline{\mathcal{M}} := \overline{\mathcal{M}}(\mathscr{A})$, $\overline{\mathcal{M}}_{\geqslant 0} := \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$, $\mathcal{M} := \mathcal{M}(\mathscr{A})$, and $\mathcal{M}_{\geqslant 0} := \mathcal{M}_{\geqslant 0}(\mathscr{A})$. If $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\overline{\mathcal{M}}$, then $\sup_{n\in\mathbb{N}} f_n$, $\inf_{n\in\mathbb{N}} f_n$, $f_\star := \liminf_{n\to\infty} f_n$, and $f^\star := \limsup_{n\to\infty} f_n$ belong also to $\overline{\mathcal{M}}$ (see lecture EWS).

(d) A real map $f \in \mathcal{M}(\mathscr{A})$ assuming only finitely many values is called *simple* or *elementary*. If $f \in \mathcal{M}(\mathscr{A})$ is simple then there is an $n \in \mathbb{N}$ and mutually disjoint measurable sets $(A_j)_{i\in[\![n]\!]}$ in $\mathscr{A}$ as well as numbers $(a_j)_{i\in[\![n]\!]}$ in $\mathbb{R}$ such that $f = \sum_{i\in[\![n]\!]} a_j \mathbb{1}_{A_j}$. We denote by $\mathcal{M}^{\text{sim}}(\mathscr{A})$ and $\mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$ the set of all simple, respectively, positive simple functions on $(\Omega, \mathscr{A})$. If $f = \sum_{i\in[\![n]\!]} a_j \mathbb{1}_{A_j}$ and $f = \sum_{i\in[\![m]\!]} b_j \mathbb{1}_{B_j}$ are two representations of $f \in \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$, then $\sum_{i\in[\![n]\!]} a_j \mu(A_j) = \sum_{i\in[\![m]\!]} b_j \mu(B_j)$ (check it!).

(e) Let $f \in \overline{\mathcal{M}}_{\geqslant 0}$ be positive numerical. Then there exists an isotone sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{M}^{\text{sim}}_{\geqslant 0}$ such that $f_n \uparrow f$ (see lecture EWS). □

§02.02 **Theorem**. *For each measure μ on a measurable space $(\Omega, \mathscr{A})$ we call integral with respect to μ the uniquely determined functional $\mathbb{I}_\mu : \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) \to \overline{\mathbb{R}}_{\geqslant 0}$ satisfying the following properties:*

(I1) $\mathbb{I}_\mu(af + bg) = a\mathbb{I}_\mu(f) + b\mathbb{I}_\mu(g)$ *for all* $f, g \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ *and* $a, b \in \mathbb{R}_{\geqslant 0}$, (linearity)

(I2) $\mathbb{I}_\mu(f_n) \uparrow \mathbb{I}_\mu(f)$ *for all* $(f_n)_{n\in\mathbb{N}} \uparrow f$ *in* $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$, (monotone convergence)

(I3) $\mathbb{I}_\mu(\mathbb{1}_A) = \mu(A)$ *for all* $A \in \mathscr{A}$. (normed)

For each $f \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ we call $\int f\,\mathrm{d}\mu := \mathbb{I}_\mu(f)$ *the integral of f with respect to μ. For $A \in \mathscr{A}$ we write shortly* $\int_A f\,\mathrm{d}\mu := \int (f\mathbb{1}_A)\,\mathrm{d}\mu$. *$f$ is called μ-integrable, if $\int f\,\mathrm{d}\mu \in \mathbb{R}_{\geqslant 0}$.* □

§02.03 **Proof** of Theorem §02.02. The theorem summarises the main result of this section; its proof takes place in several steps. We first show in Theorem §02.05 the uniqueness result and then explicitly state in Theorem §02.09 a functional $\mathbb{I}_\mu : \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) \to \overline{\mathbb{R}}_{\geqslant 0}$ for which we verify the required conditions (I1)-(I3). In summary, we then show therewith in Theorem §02.09 the existence result. □

§02.04 **Notation**. For $f \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ and $A \in \mathscr{A}$ we write shortly $\mu(f) := \int f\,\mathrm{d}\mu = \int_\Omega f(\omega)\mu(\mathrm{d}\omega)$ as well as $\mu(f\mathbb{1}_A) = \int_A f\,\mathrm{d}\mu = \int_A f(\omega)\mu(\mathrm{d}\omega)$. □

§02.05 **Uniqueness theorem**. *The integral is uniquely determined.*

§02.06 **Proof** of Theorem §02.05. is given in the lecture. □

Reminder §02.01 (e) allows the following definition to be made since the defined value $\widetilde{\mathbb{I}}_\mu(f)$ does not depend on the chosen representation of f.

§02.07 **Lemma**. *The map $\widetilde{\mathbb{I}}_\mu : \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A}) \to \overline{\mathbb{R}}_{\geqslant 0}$ given by*

$$f = \sum_{i\in[\![n]\!]} a_j \mathbb{1}_{A_j} \mapsto \widetilde{\mathbb{I}}_\mu(f) := \sum_{i\in[\![n]\!]} a_j \mu(A_j).$$

is normed, positive, linear and monotone:

(i) $\widetilde{\mathbb{I}}_\mu(\mathbb{1}_A) = \mu(A)$ *for every* $A \in \mathscr{A}$, (normed)

(ii) $\widetilde{\mathbb{I}}_\mu(af + bg) = a\widetilde{\mathbb{I}}_\mu(f) + b\widetilde{\mathbb{I}}_\mu(g)$ *for all* $f, g \in \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$ *and* $a, b \in \mathbb{R}_{\geqslant 0}$, (linearity)

(iii) $\widetilde{\mathbb{I}}_\mu(f) \leqslant \widetilde{\mathbb{I}}_\mu(g)$ *for all* $f, g \in \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$ *with* $f \leqslant g$. (monotonicity).

§02.08 **Proof** of Lemma §02.07. Exercise. □

§02.09 **Existence theorem.** *The functional* $\mathbb{I}_\mu : \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) \to \overline{\mathbb{R}}_{\geqslant 0}$ *with*

$$f \mapsto \mathbb{I}_\mu(f) := \sup\left\{\widetilde{\mathbb{I}}_\mu(g) : g \in \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A}), g \leqslant f\right\}$$

is an integral with respect to μ*, that is, it shares the properties* (I1)-(I3) *in Theorem §02.02:*

(i) $\mathbb{I}_\mu(\mathbb{1}_A) = \mu(A)$ *for every* $A \in \mathscr{A}$, (normed)

(ii) $\mathbb{I}_\mu(f) \leqslant \mathbb{I}_\mu(g)$ *for all* $f, g \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ *with* $f \leqslant g$, (monotonicity)

(iii) $\mathbb{I}_\mu(f_n) \uparrow \mathbb{I}_\mu(f)$ *for all* $(f_n)_{n\in\mathbb{N}} \uparrow f$ *in* $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$. (monotone convergence)

(iv) $\mathbb{I}_\mu(af + bg) = a\mathbb{I}_\mu(f) + b\mathbb{I}_\mu(g)$ *for all* $f, g \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ *and* $a, b \in \overline{\mathbb{R}}_{\geqslant 0}$ (linearity)

(with convention $\infty \cdot 0 = 0$*).*

§02.10 **Proof** of Theorem §02.09. is given in the lecture. □

§02.11 **Remark.** By Lemma §02.07 (iii) we have the identity $\mathbb{I}_\mu(f) = \widetilde{\mathbb{I}}_\mu(f)$ for any $f \in \mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$. Hence $\mathbb{I}_\mu$ is an extension of the map $\widetilde{\mathbb{I}}_\mu$ from $\mathcal{M}^{\text{sim}}_{\geqslant 0}(\mathscr{A})$ to the set of positive numerical functions $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$. □

§02.12 **Comment.** A measurable partition $\mathcal{P} := \{A_i : i \in \mathcal{I}\} \subseteq \mathscr{A}_{\setminus\emptyset}$ of Ω is finite, if $|\mathcal{I}| \in \mathbb{N}$, and hence $\emptyset \neq A \in \mathscr{A}$ for each $A \in \mathcal{P}$. If we set $\mathscr{P} := \{\mathcal{P} \subseteq \mathscr{A}_{\setminus\emptyset} : \mathcal{P} \text{ finite, measurable partition of } \Omega\}$, then the functional $\mathbb{I}_\mu : \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) \to \overline{\mathbb{R}}_{\geqslant 0}$ given by (with convention $\infty \cdot 0 = 0$)

$$f \mapsto \mathbb{I}_\mu(f) := \sup\Big\{\sum_{A\in\mathcal{P}}\big(\inf_{\omega\in A} f(\omega)\big)\mu(A) : \mathcal{P} \in \mathscr{P}\Big\}$$

shares also the properties (I1)-(I3) in Theorem §02.02, and hence it is an alternative but equivalent representation of the uniquely determined integral with respect to μ. □

§02.13 **Notation.** For arbitrary measures $\mu, \nu \in \mathfrak{M}(\mathscr{A})$ we write $\nu \leqslant \mu$ if $\nu(A) \leqslant \mu(A)$ for all $A \in \mathscr{A}$. Evidently, $\nu \leqslant \mu$ and $\mu \leqslant \nu$ imply together $\mu = \nu$. □

§02.14 **Lemma** *(Properties). Let* $(\Omega, \mathscr{A}, \mu)$ *be an arbitrary measure space and let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

(i) *(Fatou's lemma)* $\mu(\liminf_{n\to\infty} f_n) = \int\big(\liminf_{n\to\infty} f_n\big)\,\mathrm{d}\mu \leqslant \liminf_{n\to\infty}\int f_n\,\mathrm{d}\mu = \liminf_{n\to\infty}\mu(f_n)$ *and in particular* $\mu\big(\liminf_{n\to\infty} A_n\big) \leqslant \liminf_{n\to\infty}\mu(A_n)$ *for every sequence* $(A_n)_{n\in\mathbb{N}}$ *of sets in* $\mathscr{A}$*. If* $\mu \in \mathfrak{M}_{\mathrm{f}}(\mathscr{A})$ *is finite, then also* $\limsup_{n\to\infty}\mu(A_n) \leqslant \mu\big(\limsup_{n\to\infty} A_n\big)$.

(ii) $\sum_{n\in\mathbb{N}} f_n \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ *and* $\mu(\sum_{n\in\mathbb{N}} f_n) = \int\big(\sum_{n\in\mathbb{N}} f_n\big)\,\mathrm{d}\mu = \sum_{n\in\mathbb{N}}\int f_n\,\mathrm{d}\mu = \sum_{n\in\mathbb{N}}\mu(f_n)$.

Let in addition $f, g \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

(iii) $f = 0$ μ-a.e. *if and only if* $\mu(f) = \int f\,\mathrm{d}\mu = 0$*. If* $\mu(f) \in \mathbb{R}_{\geqslant 0}$ *then* $f \in \mathbb{R}_{\geqslant 0}$ μ-a.e. *and the restriction of* μ *on* $\{f \neq 0\}$ *is a* σ*-finite measure.*

(iv) *The set function* $f\mu : \mathscr{A} \to \overline{\mathbb{R}}_{\geqslant 0}$ *with* $A \mapsto f\mu(A) := \mu(\mathbb{1}_A f) = \int(\mathbb{1}_A f)\,\mathrm{d}\mu$ *is a measure on* $(\Omega, \mathscr{A})$*. For all* $A \in \mathscr{A}$ *with* $\mu(A) = 0$ *we have* $f\mu(A) = 0$.

(v) *If* $f \leqslant g$ *(respectively* $f = g$*)* μ-a.e. *then* $f\mu \leqslant g\mu$ *(respectively* $f\mu = g\mu$*). The converse holds, if* (c1) f *is* μ*-integrable, or* (c2) $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$*, or* (c3) $g\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ σ*-finite.*
In particular, $\mu(f) = \int f\,\mathrm{d}\mu \leqslant \int g\,\mathrm{d}\mu = \mu(g)$ *(respectively,* $\mu(f) = \mu(g)$*).*

(vi) $\mu \in \mathfrak{M}(\mathscr{A})$ *is* σ*-finite if and only if there is* $h \in \mathcal{M}_{(0,1]}(\mathscr{A})$ *with* $\mu(h) \in \mathbb{R}_{\geqslant 0}$ *(*μ*-integrable). In particular, for each* σ*-finite* $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ *there exists* $h \in \mathcal{M}(\mathscr{A})$ *with* $h \in \mathbb{R}_{>0}$ μ-a.e. *such that* $h\mu \in \mathfrak{M}_{\mathrm{f}}(\mathscr{A})$ *is finite and* $h\mu$ *shares the same null-sets as* μ.

(vii) $\sum_{n\in\mathbb{N}} \mu(\{f \geqslant n\}) \leqslant \mu(f) \leqslant \sum_{n\in\mathbb{N}_0} \mu(\{f > n\})$ *and* $\mu(f) = \int_0^\infty \mu(\{f \geqslant t\})\mathrm{d}t$ *for every* $f \in \mathcal{M}(\mathscr{A})$ *with* $f \in \mathbb{R}_{\geqslant 0}$ μ-a.e.

§02.15 **Proof** of Lemma §02.14. is given in the lecture. □

§02.16 **Definition**. Let $\mu \in \mathfrak{M}(\mathscr{A})$ be a measure on $(\Omega, \mathscr{A})$ and let $\mathfrak{f} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$. Define the measure $\nu \in \mathfrak{M}(\mathscr{A})$ by $\nu(A) := \mu(\mathbb{1}_A \mathfrak{f})$ for $A \in \mathscr{A}$. We say that $\mathfrak{f}\mu := \nu$ has the *density* $\mathrm{d}\nu/\mathrm{d}\mu := \mathfrak{f}$ with respect to μ, or briefly *μ-density*. □

§02.17 **Lemma** *(Properties)*. *Let* $(\Omega, \mathscr{A}, \mu)$ *be a measure space and let* $\nu := \mathfrak{f}\mu \in \mathfrak{M}(\mathscr{A})$ *admit the density* $\mathrm{d}\nu/\mathrm{d}\mu = \mathfrak{f} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

(i) $\nu(g) = \int g\,\mathrm{d}\nu = \int (g\mathfrak{f})\,\mathrm{d}\mu = \mu(g\mathfrak{f}) = \mathfrak{f}\mu(g)$ *for every* $g \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

(ii) $\rho = \mathfrak{q}\nu = \mathfrak{q}(\mathfrak{f}\mu) = (\mathfrak{q}\mathfrak{f})\mu$ *for every* $\rho := \mathfrak{q}\nu \in \mathfrak{M}(\mathscr{A})$ *with* $\mathfrak{q} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

(iii) *If* $\nu \in \mathfrak{M}_\sigma(\mathscr{A})$ *or* $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ *is σ-finite then the μ-density* $\mathrm{d}\nu/\mathrm{d}\mu = \mathfrak{f}$ *of* ν *is unique up to equality μ-almost everywhere.*

(iv) *If* $\nu \in \mathfrak{M}_\sigma(\mathscr{A})$ *is σ-finite, then* $\mathrm{d}\nu/\mathrm{d}\mu = \mathfrak{f} \in \mathbb{R}_{\geqslant 0}$ μ-a.e.. *The converse holds, if* $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$.

§02.18 **Proof** of Lemma §02.17. is given in the lecture. □

§02.19 **Notation**. If $f \in \overline{\mathcal{M}}(\mathscr{A})$ is numerical then $f^+ := f \vee 0$, $f^- := (-f)^+$, $|f| = f^+ + f^- \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ are positive numerical. □

§02.20 **Definition**. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $f \in \overline{\mathcal{M}}(\mathscr{A})$ be numerical.

(a) If f^+ or f^- is μ-integrable, that is, $\mu(f^+) \wedge \mu(f^-) \in \mathbb{R}_{\geqslant 0}$, then we define the *integral*

$$\mu(f) := \int f\,\mathrm{d}\mu := \int f^+\,\mathrm{d}\mu - \int f^-\,\mathrm{d}\mu = \mu(f^+) - \mu(f^-)$$

of f with respect to μ where we use the usual conventions $\infty + x = \infty$ and $-\infty + x = -\infty$ for all $x \in \mathbb{R}$. In this case f is called *μ-quasiintegrable*. The integral of f is not defined, if $\mu(f^+) = \infty = \mu(f^-)$.

(b) If $\mu(|f|) \in \mathbb{R}_{\geqslant 0}$, that is, $\mu(f^+) \vee \mu(f^-) \in \mathbb{R}_{\geqslant 0}$, then f is called *μ-integrable*. The set of all μ-integrable numerical functions is denoted by

$$\mathcal{L}_1 := \mathcal{L}_1(\mu) := \mathcal{L}_1(\Omega, \mathscr{A}, \mu) := \left\{f \in \overline{\mathcal{M}}(\mathscr{A}) \colon \mu(|f|) \in \mathbb{R}_{\geqslant 0}\right\}.$$

(c) For $p \in \mathbb{R}_{>0}$ define

$$\|f\|_{\mathcal{L}_p} := \left(\mu(|f|^p)\right)^{1/p} \quad \text{and} \quad \|f\|_{\mathcal{L}_\infty} := \inf\left\{x \in \mathbb{R}_{\geqslant 0} \colon \mu(\{|f| > x\}) = 0\right\}.$$

For $p \in \overline{\mathbb{R}}_{>0}$ a function f is called *$\mathcal{L}_p$-integrable* if $\|f\|_{\mathcal{L}_p} \in \mathbb{R}_{\geqslant 0}$. The vector space of all $\mathcal{L}_p$-integrable functions we denote by

$$\mathcal{L}_p := \mathcal{L}_p(\mu) := \mathcal{L}_p(\Omega, \mathscr{A}, \mu) := \left\{f \in \overline{\mathcal{M}}(\mathscr{A}) \colon \|f\|_{\mathcal{L}_p} \in \mathbb{R}_{\geqslant 0}\right\}.$$

For $p \in \overline{\mathbb{R}}_{\geqslant 1}$, the map $\|\cdot\|_{\mathcal{L}_p}$ is a seminorm on $\mathcal{L}_p(\mu)$ (see Subsection §02|03 below), that is, for all $f, g \in \mathcal{L}_p(\mu)$ and $a \in \mathbb{R}$, (s1) $\|af\|_{\mathcal{L}_p} = |a|\|f\|_{\mathcal{L}_p}$, (s2) $\|f + g\|_{\mathcal{L}_p} \leqslant \|f\|_{\mathcal{L}_p} + \|g\|_{\mathcal{L}_p}$, (s3) $\|f\|_{\mathcal{L}_p} \in \mathbb{R}_{\geqslant 0}$ and $\|f\|_{\mathcal{L}_p} = 0$ if $f = 0$ μ-a.e.

(d) The map $\langle\cdot,\cdot\rangle_{\mathcal{L}_2} : \mathcal{L}_2(\mu) \times \mathcal{L}_2(\mu) \to \mathbb{R}$ with $(f, g) \mapsto \langle f, g\rangle_{\mathcal{L}_2} := \mu(fg)$ is a positive semidefinite symmetric bilinearform. □

§02.21 **Lemma** *(Properties).* *Let $f, g \in \mathcal{L}_1(\Omega, \mathscr{A}, \mu)$.*

(i) *If $a, b \in \mathbb{R}$, then $af + bg \in \mathcal{L}_1(\mu)$ and $\int (af + bg)\,\mathrm{d}\mu = a \int f\,\mathrm{d}\mu + b \int g\,\mathrm{d}\mu$.* *(linearity)*

(ii) *Let $h \in \overline{\mathcal{M}}(\mathscr{A})$. If $h = f$ μ-a.e., then $h \in \mathcal{L}_1(\mu)$ and $\int h\,\mathrm{d}\mu = \int f\,\mathrm{d}\mu$. If $|h| \leqslant |g|$ μ-a.e. then $h \in \mathcal{L}_1(\mu)$.*

(iii) *If $f \leqslant g$ μ-a.e., then $\mu(f) \leqslant \mu(g)$.* *(monotonicity)*
In particular, if $f \in \overline{\mathbb{R}}_{\geqslant 0}$ μ-a.e., then $\mu(f) \in \mathbb{R}_{\geqslant 0}$. *(positive)*

(iv) $|\mu(f)| \leqslant \mu(|f|)$. *(triangle inequality)*

(v) *$f = 0$ μ-a.e. if and only if $\mu(f\mathbb{1}_A) = 0$ for all $A \in \mathscr{A}$.*

(vi) *If $\mu \in \mathfrak{M}_{\mathrm{f}}(\mathscr{A})$ is finite and $h \in \mathcal{M}(\mathscr{A})$ is bounded, hence $\|h\|_\infty := \sup_{\omega\in\Omega} |h(\omega)| \in \mathbb{R}_{\geqslant 0}$, then $h \in \mathcal{L}_1(\mu)$.*

(vii) *If $\mu, \nu \in \mathfrak{M}(\mathscr{A})$ then $h \in \mathcal{L}_1(\mu + \nu)$ if and only if $h \in \mathcal{L}_1(\mu) \cap \mathcal{L}_1(\nu)$. In this case, $(\mu + \nu)(h) = \mu(h) + \nu(h)$.*

(viii) *If $\nu = \mathfrak{f}\mu$ with $\mathrm{d}\nu/\mathrm{d}\mu = \mathfrak{f} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ then $g \in \overline{\mathcal{M}}(\mathscr{A})$ is ν-(quasi)integrable if and only if $g\mathfrak{f} \in \overline{\mathcal{M}}(\mathscr{A})$ is μ-(quasi)integrable. In this case $\nu(g) = \mu(g\mathfrak{f}) = \int (g\mathfrak{f})\,\mathrm{d}\mu = \int g\,\mathrm{d}(\mathfrak{f}\mu) = \int g\,\mathrm{d}\nu$.*

§02.22 **Proof** of Lemma §02.21. is given in the lecture. □

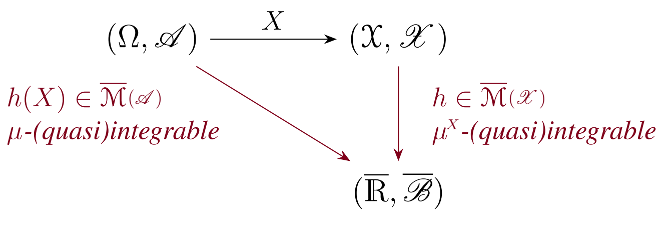
§02.23 **Corollary** *(Properties).* *Let $f, g \in \overline{\mathcal{M}}(\mathscr{A})$ and $\mu \in \mathfrak{M}(\mathscr{A})$.*

(i) *Let $p \in \mathbb{R}_{>0}$. $f \in \mathcal{L}_p(\mu)$ if and only if $|f|^p \in \mathcal{L}_1(\mu)$. Moreover, if $f \in \mathcal{L}_\infty(\mu)$ then $\mu\big(\{|f| > \|f\|_{\mathcal{L}_\infty}\}\big) = 0$.*

(ii) *Let $p \in \overline{\mathbb{R}}_{>0}$. $\|f\|_{\mathcal{L}_p} = 0$ if and only if $f = 0$ μ-a.e.. If $a \in \mathbb{R}$ then $\|af\|_{\mathcal{L}_p} = |a|\|f\|_{\mathcal{L}_p}$. If $f \in \mathcal{L}_p(\mu)$ and $f = g$ μ-a.e., then $|f| \in \mathbb{R}_{\geqslant 0}$ μ-a.e. and $\|f\|_{\mathcal{L}_p} = \|g\|_{\mathcal{L}_p}$.*

§02.24 **Proof** of Corollary §02.23. Exercise. □

§02.25 **Lemma** *(Image measure).* *Let $(\Omega, \mathscr{A})$ and $(\mathcal{X}, \mathscr{X})$ be measurable spaces, let $\mu \in \mathfrak{M}(\mathscr{A})$ be a measure and let $X \in \mathcal{M}(\mathscr{A}, \mathscr{X})$ be measurable. Let $\mu^X := \mu \circ X^{-1} \in \mathfrak{M}(\mathscr{X})$ be the image measure on $(\mathcal{X}, \mathscr{X})$ of μ under the map X. If $h \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{X})$ then $\mu(h(X)) = \mu^X(h)$. Consequently, $h \in \overline{\mathcal{M}}(\mathscr{X})$ is μ^X-(quasi)integrable if and only if $h(X) \in \overline{\mathcal{M}}(\mathscr{A})$ is μ-(quasi)integrable. In this case, $\mu(h(X)) = \mu^X(h)$.*



In particular, if X is a random variable on $(\Omega, \mathscr{A}, \mathbb{P})$, then

$$\int_{\mathcal{X}} h(x)\mathbb{P}^X(\mathrm{d}x) = \int h\,\mathrm{d}\mathbb{P}^X = \mathbb{P}^X(h) = \mathbb{P}(h(X)) = \int h(X)\,\mathrm{d}\mathbb{P} = \int_\Omega h(X(\omega))\mathbb{P}(\mathrm{d}\omega).$$

§02.26 **Proof** of Lemma §02.25. is given in the lecture. □

§02|02 Convergence criteria

§02.27 **Definition**. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space. We say that a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{M}(\mathscr{A})$ converges to $f \in \overline{\mathcal{M}}(\mathscr{A})$

μ-almost everywhere (μ-a.e.), symbolically $f_n \xrightarrow{\mu\text{-a.e.}} f$, if $\limsup_{n\to\infty}|f_n - f| = 0$ μ-a.e., that is, there exists a μ-null set $N \in \mathscr{A}$ such that $\lim_{n\to\infty}|f_n(\omega) - f(\omega)| = 0$ for any $\omega \in N^c := \Omega \setminus N$.

μ-almost complete (μ-a.c.), symbolically $f_n \xrightarrow{\mu\text{-a.c.}} f$, if $\sum_{n\in\mathbb{N}} \mu\big(\{|f_n - f| > \varepsilon\} \cap A\big) \in \mathbb{R}_{\geqslant 0}$ for every $A \in \mathscr{A}$ with $\mu(A) \in \mathbb{R}_{\geqslant 0}$ and for every $\varepsilon \in \mathbb{R}_{>0}$.

in μ-measure (or, briefly, in measure), symbolically $f_n \xrightarrow{\mu} f$, if $\lim_{n\to\infty} \mu\big(\{|f_n - f| > \varepsilon\} \cap A\big) = 0$ for every $A \in \mathscr{A}$ with $\mu(A) \in \mathbb{R}_{\geqslant 0}$ and for every $\varepsilon \in \mathbb{R}_{>0}$.

in $\mathcal{L}_p(\mu)$ (or in p-th μ-mean) for $p \in \overline{\mathbb{R}}_{>0}$, symbolically $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$, if $(f_n)_{n\in\mathbb{N}}$ and f in $\mathcal{L}_p(\mu)$ such that $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{L}_p} = 0$.

If μ is a probability measure, then convergence in μ-measure is also called convergence *in probability*. Sometimes we write briefly $f_n \xrightarrow{\text{a.e.}} f$, $f_n \xrightarrow{\text{a.c.}} f$ or $f_n \xrightarrow{\mathcal{L}_p} f$ if the underlying measure emerges from the context. □

§02.28 **Remark**. Convergence in $\mathcal{L}_p(\mu)$ and convergence μ-almost everywhere evidently determine the limit up to equality μ-almost everywhere. This also applies to convergence in μ-measure, if $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is σ-finite. Indeed, if $f_n \xrightarrow{\mu} f$ and $f_n \xrightarrow{\mu} g$ then for every $\varepsilon \in \mathbb{R}_{>0}$ and $A \in \mathscr{A}$ with $\mu(A) \in \mathbb{R}_{\geqslant 0}$ (since $|f - g| \leqslant |f - f_n| + |g - f_n|$)

$$\mu(\{|f-g| > \varepsilon\} \cap A) \leqslant \mu(\{|f - f_n| > \varepsilon/2\} \cap A) + \mu(\{|g - f_n| > \varepsilon/2\} \cap A) \xrightarrow{n\to\infty} 0.$$

and hence $\mu(\{|f - g| > \varepsilon\} \cap A) = 0$. Therefore, we have $\mu(\{f \neq g\} \cap A) = 0$ making use of $\{f \neq g\} \cap A = \bigcup_{k\in\mathbb{N}}\{|f - g| > 1/k\} \cap A$. Selecting $A_n \uparrow \Omega$ with $\mu(A_n) \in \mathbb{R}_{\geqslant 0}$ (since $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$) implies $f = g$ μ-a.e.. If $\mu \in \mathfrak{M}_f(\mathscr{A})$ is finite, then $\lim_{n\to\infty} \mu(\{|f_n - f| > \varepsilon\}) = 0$ for every $\varepsilon \in \mathbb{R}_{>0}$ and $f_n \xrightarrow{\mu} f$ are equivalent. The last statement does not apply, if $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is σ-finite. For instance, on $(\mathbb{N}, 2^{\mathbb{N}}, \zeta_{\mathbb{N}})$ (see Example §01.18 (c) for the counting measure $\zeta_{\mathbb{N}}$) for $A_n := \mathbb{N}_{\geqslant n}$, $n \in \mathbb{N}$, we have $\{\mathbb{1}_{A_n} > \varepsilon\} = A_n$ for every $\varepsilon \in (0,1)$ and $\{\mathbb{1}_{A_n} > \varepsilon\} = \emptyset$ for every $\varepsilon \in \mathbb{R}_{\geqslant 1}$. Since $A_n \downarrow \emptyset$, and hence $\zeta_{\mathbb{N}}(A_n \cap A) \downarrow 0$ for each $A \in \mathscr{A}$ with $\zeta_{\mathbb{N}}(A) \in \mathbb{R}_{\geqslant 0}$ (upper semicontinuous), we evidently have $\mathbb{1}_{A_n} \xrightarrow{\zeta_{\mathbb{N}}} 0$. On the other hand side, for each $\varepsilon \in (0,1)$ we have $\zeta_{\mathbb{N}}(\{\mathbb{1}_{A_n} > \varepsilon\}) = \zeta_{\mathbb{N}}(A_n) = \infty$ for all $n \in \mathbb{N}$. □

§02.29 **Lemma**. *Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space.*

(i) *(Monotone convergence) Let $f \in \overline{\mathcal{M}}(\mathscr{A})$ and let $f_n \in \mathcal{L}_1(\mu)$, $n \in \mathbb{N}$. Assume $f_n \uparrow f$ μ-a.e. Then $\mu(f_n) \uparrow \mu(f)$ where both sides can equal $+\infty$.*

(ii) *(Dominated convergence) Let $(f_n)_{n\in\mathbb{N}}$ in $\overline{\mathcal{M}}(\mathscr{A})$ be μ-a.e. convergent. Assume $\sup_{n\in\mathbb{N}}|f_n| \leqslant g$ μ-a.e. with $g \in \mathcal{L}_1(\mu)$. Then there exists $f \in \mathcal{M}(\mathscr{A})$ with $f_n \xrightarrow{\mu\text{-a.e.}} f$, $(f_n)_{n\in\mathbb{N}}$ and f belong to $\mathcal{L}_1(\mu)$ and $\lim_{n\to\infty} \mu((|f - f_n|)) = 0$ as well as $\lim_{n\to\infty} \mu(f_n) = \mu(f)$. If $g \in \mathcal{L}_p(\mu)$ for $p \in \mathbb{R}_{\geqslant 1}$, then $(f_n)_{n\in\mathbb{N}}$ and f belong to $\mathcal{L}_p(\mu)$ and $\lim_{n\to\infty}\|f_n - f\|_{\mathcal{L}_p} = 0$.*

(iii) *(Scheffé's theorem) Let $f, f_n \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$, $n \in \mathbb{N}$, be μ-integrable. Assume $f_n \xrightarrow{\mu\text{-a.e.}} f$ and $\mu(f_n) \xrightarrow{n\to\infty} \mu(f)$, then $f_n \xrightarrow{\mathcal{L}_1(\mu)} f$.*

(iv) *(Theorem of Riesz) Let $f, f_n \in \mathcal{L}_p(\mu)$, $n \in \mathbb{N}$, with $p \in \mathbb{R}_{\geqslant 1}$ Assume $f_n \xrightarrow{\mu\text{-a.e.}} f$. $\mu(|f_n|^p) \xrightarrow{n\to\infty} \mu(|f|^p)$ if and only if $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$.*

(v) *Let $f, f_n \in \mathcal{M}(\mathscr{A})$, $n \in \mathbb{N}$. Then the following implications hold:*

$$f_n \xrightarrow{\mu\text{-a.e.}} f \Longrightarrow f_n \xrightarrow{\mu} f \Longleftarrow f_n \xrightarrow{\mathcal{L}_p(\mu)} f.$$

If $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is σ-finite, then we also have $f_n \xrightarrow{\mu\text{-a.c.}} f \Longrightarrow f_n \xrightarrow{\mu\text{-a.e.}} f$. Moreover, $f_n \xrightarrow{\mu} f$ if and only if for any subsequent of $(f_n)_{n\in\mathbb{N}}$ there exists a sub-subsequence that converges to f μ-almost everywhere.

§02.30 **Proof** of Lemma §02.29. is given in the lecture. □

§02.31 **Reminder**. A sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{L}_p(\mu)$ is called ($\mathcal{L}_p(\mu)$-)Cauchy sequence, if for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $n_\circ \in \mathbb{N}$ such that $\|f_n - f_m\|_{\mathcal{L}_p(\mu)} \leqslant \varepsilon$ for all $m, n \in \mathbb{N}_{\geqslant n_\circ}$, symbolically $\lim_{n,m\to\infty}\|f_n - f_m\|_{\mathcal{L}_p(\mu)} = 0$. Keep in mind that every $\mathcal{L}_p(\mu)$ convergent sequence by applying Minkowski's inequality (see Lemma §02.50 (iii)) is also a $\mathcal{L}_p(\mu)$-Cauchy sequence. □

§02.32 **Lemma**. *Let $p \in \overline{\mathbb{R}}_{\geqslant 1}$ and let $(f_n)_{n\in\mathbb{N}}$ be a $\mathcal{L}_p(\mu)$-Cauchy sequence. Then there exists $f \in \mathcal{L}_p(\mu)$ such that $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$ and there exists a subsequence of $(f_n)_{n\in\mathbb{N}}$ that converges μ-a.e. to f.*

§02.33 **Proof** of Lemma §02.32. is given in the lecture. □

§02.34 **Corollary**. *Let $p \in \overline{\mathbb{R}}_{\geqslant 1}$ and let $(f_n)_{n\in\mathbb{N}}$ be a $\mathcal{L}_p(\mu)$-Cauchy sequence that converges μ-a.e. to $f \in \mathcal{M}(\mathscr{A})$. Then f belongs to $\mathcal{L}_p(\mu)$ and $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$.*

§02.35 **Proof** of Corollary §02.34. is given in the lecture. □

§02.36 **Preliminaries**. Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space, let $p \in \mathbb{R}_{\geqslant 1}$ and let $f \in \overline{\mathcal{M}}(\mathscr{A})$. f is μ-integrable if and only if for every $\varepsilon \in \mathbb{R}_{>0}$ there exists $g \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ such that $\mu(|f|\mathbb{1}_{\{|f|\geqslant g\}}) \leqslant \varepsilon$ or in equal $\inf\{\mu(|f|\mathbb{1}_{\{|f|\geqslant g\}}) : g \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})\} = 0$. Assume $\mu(|f|) \in \mathbb{R}_{\geqslant 0}$. Setting $g := 2|f| \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ we evidently have $\{|f| \geqslant g\} = \{f = 0\} \cup \{|f| = \infty\}$ and hence applying Corollary §02.23 (ii) also $\mu(|f|\mathbb{1}_{\{|f|\geqslant g\}}) = 0$. We obtain the converse by exploiting $\mu(|f|) = \mu(|f|\mathbb{1}_{\{|f|\geqslant g\}}) + \mu(|f|\mathbb{1}_{\{|f|<g\}}) \leqslant \varepsilon + \mu(g) \in \mathbb{R}_{>0}$, which in turn implies $\mu(|f|) \in \mathbb{R}_{\geqslant 0}$. □

§02.37 **Definition**. A class of functions $\mathcal{F} \subseteq \mathcal{L}_1(\mu)$ is called *uniformly μ-integrable* if

$$\inf\Big\{\sup_{f\in\mathcal{F}} \mu\big(|f|\mathbb{1}_{\{|f|\geqslant g\}}\big) : g \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})\Big\} = 0.$$

If $\mu \in \mathfrak{M}_f(\mathscr{A})$ is finite, then uniform μ-integrability is equivalent to the condition:

$$\inf\Big\{\sup_{f\in\mathcal{F}} \mu\big(|f|\mathbb{1}_{\{|f|\geqslant a\}}\big) : a \in \mathbb{R}_{\geqslant 0}\Big\} = 0.$$ □

§02.38 **Remark**.

(a) Let $\mathcal{F}$ be uniformly μ-integrable and let $\varepsilon \in \mathbb{R}_{>0}$. A function $g \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ is called *ε-majorant* if $\sup_{f\in\mathcal{F}} \mu\big(|f|\mathbb{1}_{\{|f|\geqslant g\}}\big) \leqslant \varepsilon$. Evidently, there exists a ε-majorant g for $\mathcal{F}$ and every $h \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ with $h \geqslant g$ is also a ε-majorant for $\mathcal{F}$.

(b) A family $(f_i)_{i\in\mathcal{I}}$ in $\overline{\mathcal{M}}(\mathscr{A})$ is called *uniformly μ-integrable* if the class $\{f_i : i \in \mathcal{I}\}$ is.

(c) Let $\mathcal{F}_i$, $i \in [\![n]\!]$, be finitly many uniformly μ-integrable classes in $\overline{\mathcal{M}}(\mathscr{A})$. Then their union $\mathcal{F} := \cup_{i\in[\![n]\!]}\mathcal{F}_i$ is also uniformly μ-integrable. Indeed, for every $\varepsilon \in \mathbb{R}_{>0}$ and ε-majorant g_i for $\mathcal{F}_i$, $i \in [\![n]\!]$, the function $g_1 \vee \cdots \vee g_n$ is a ε-majorant for $\mathcal{F}$.

(d) Let $\mathcal{F} \subseteq \overline{\mathcal{M}}(\mathscr{A})$ and let $g \in \mathcal{L}_p(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ satisfy $|f| \leqslant g$ μ-a.e. for all $f \in \mathcal{F}$. Then $\mathcal{F}^p := \{|f|^p : f \in \mathcal{F}\}$ is uniformly μ-integrable. For $\varepsilon \in \mathbb{R}_{>0}$ every ε-majorant h for $\{g^p\}$ is a ε-majorant for $\mathcal{F}^p$, since $\mu\big(|f|^p\mathbb{1}_{\{|f|^p>h\}}\big) \leqslant \mu\big(|g|^p\mathbb{1}_{\{|g|^p>h\}}\big) \leqslant \varepsilon$ for all $f \in \mathcal{F}$. □

§02.39 **Lemma**. *Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space.*

(i) *If $\mathcal{F} \subseteq \mathcal{L}_1(\mu)$ is a finite set then $\mathcal{F}$ is uniformly μ-integrable.*

(ii) *If $\mathcal{F}, \mathcal{G} \subseteq \mathcal{L}_1(\mu)$ are uniformly μ-integrable, then $\{f + g \colon f \in \mathcal{F}, g \in \mathcal{G}\}$, $\{f - g \colon f \in \mathcal{F}, g \in \mathcal{G}\}$ and $\{|f| \colon f \in \mathcal{F}\}$ are uniformly μ-integrable.*

(iii) *If $\mathcal{F} \subseteq \mathcal{L}_1(\mu)$ is uniformly μ-integrable, and if, for any $g \in \mathcal{G} \subseteq \overline{\mathcal{M}}(\mathscr{A})$, there exists an $f \in \mathcal{F}$ with $|g| \leqslant |f|$, then $\mathcal{G} \subseteq \mathcal{L}_1(\mu)$ is also uniformly μ-integrable.*

(iv) *Let $\mu \in \mathfrak{M}_f(\mathscr{A})$ be finite, let $p \in \mathbb{R}_{>1}$ and let $\mathcal{F}$ be bounded in $\mathcal{L}_p(\mu)$, that is, $\sup\{\|f\|_{\mathcal{L}_p} \colon f \in \mathcal{F}\} \in \mathbb{R}_{\geqslant 0}$. Then $\mathcal{F}$ is uniformly μ-integrable.*

§02.40 **Proof** of Lemma §02.39. We refer to the lecture EWS / Exercise. □

§02.41 **Theorem**. *Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space. $\mathcal{F} \subseteq \overline{\mathcal{M}}(\mathscr{A})$ is uniformly μ-integrable if and only if the following two conditions hold:*

(gI1) *$\mathcal{F}$ is bounded in $\mathcal{L}_1(\mu)$, i.e. $\sup\{\mu(|f|) \colon f \in \mathcal{F}\} \in \mathbb{R}_{\geqslant 0}$.*

(gI2) *For any $\varepsilon \in \mathbb{R}_{>0}$ there are $h \in \mathcal{L}_1(\mu) \cap \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ and $\delta \in \mathbb{R}_{>0}$ such that for all $A \in \mathscr{A}$ holds the implication: $\mu(h\mathbb{1}_A) \leqslant \delta \Rightarrow \sup_{f \in \mathcal{F}} \mu(|f|\mathbb{1}_A) \leqslant \varepsilon$.*

§02.42 **Proof** of Theorem §02.41. is given in the lecture. □

§02.43 **Theorem**. *Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, let $p \in \mathbb{R}_{\geqslant 1}$ and let $(f_n)_{n \in \mathbb{N}}$ belong to $\mathcal{L}_p(\mu) \cap \mathcal{M}(\mathscr{A})$. Then (i) $(f_n)_{n \in \mathbb{N}}$ converges in $\mathcal{L}_p(\mu)$, is equivalent to (ii) $(|f_n|^p)_{n \in \mathbb{N}}$ is uniformly μ-integrable and $(f_n)_{n \in \mathbb{N}}$ converges in μ-measure.*

§02.44 **Proof** of Theorem §02.43. (i)⇒(ii) in the lecture, for the converse we refer to Bauer (1992, Theorem 21.4, p.142) □

§02.45 **Remark**. The Theorem §02.43 guarantees the existence of a $\mathcal{L}_p(\mu)$-integrable function under the possible limits in μ-measure of the sequence $(f_n)_{n \in \mathbb{N}}$. □

§02.46 **Corollary**. *Let $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ be σ-finite, let $p \in \mathbb{R}_{\geqslant 1}$ and let $(f_n)_{n \in \mathbb{N}}$ belong to $\mathcal{L}_p(\mu)$. Assume $f_n \xrightarrow{\mu} f \in \mathcal{M}(\mathscr{A})$ and $(|f_n|^p)_{n \in \mathbb{N}}$ is uniformly μ-integrable. Then $f \in \mathcal{L}_p(\mu)$ and $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$.*

§02.47 **Proof** of Corollary §02.46. is given in the lecture. □

§02.48 **Summary**. *Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space, let $p \in \overline{\mathbb{R}}_{\geqslant 1}$, and let $(f_n)_{n \in \mathbb{N}}$ belong to $\mathcal{L}_p(\mu)$. Then the following claims are equivalent:*

(i) *There is $f \in \mathcal{L}_p(\mu)$ such that $f_n \xrightarrow{\mathcal{L}_p(\mu)} f$.*

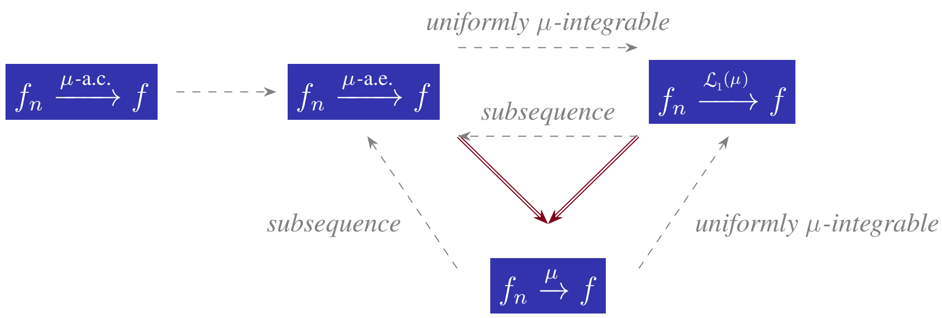
(ii) *$(f_n)_{n \in \mathbb{N}}$ is a $\mathcal{L}_p(\mu)$-Cauchy sequence, i.e. $\lim_{n,m \to \infty} \|f_n - f_m\|_{\mathcal{L}_p} = 0$.*

Assume in addition $p \in \mathbb{R}_{\geqslant 1}$ and $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is σ-finite. Then (i) and (ii) are equivalent to

(iii) *$(|f_n|^p)_{n \in \mathbb{N}}$ is uniformly μ-integrable, and there is $f \in \mathcal{M}(\mathscr{A})$ such that $f_n \xrightarrow{\mu} f$.*

The limes in (i) and in (iii) coincide.

Figure 02 [§02] Implications of convergence criteria.



The Figure 02 [§02] was created based on Klenke (2020, Abb.6.1, p.159). □

§02|03 $\mathcal{L}_p$-Spaces

§02.49 **Reminder**. For $p \in \overline{\mathbb{R}}_{>0}$ and $f, g \in \overline{\mathcal{M}}(\mathscr{A})$ we have shown that $\|f - g\|_{\mathcal{L}_p(\mu)} = 0$ if and only if $f = g$ μ-a.e.. In this case we now consider f and g as equivalent. More precisely, for each $f \in \overline{\mathcal{M}}(\mathscr{A})$ we introduce the μ-equivalence class $\{f\}_\mu := \{g \in \overline{\mathcal{M}}(\mathscr{A}) : g = f \ \mu\text{-a.e.}\}$ and hence $\{0\}_\mu = \{g \in \overline{\mathcal{M}}(\mathscr{A}) : g = 0 \ \mu\text{-a.e.}\}$. For any $p \in \overline{\mathbb{R}}_{\geqslant 1}$, $\{0\}_\mu$ is a subvector space of $\mathcal{L}_p(\mu)$. Thus formally we can build the factor space

$$\mathbb{L}_p := \mathbb{L}_p(\mu) := \mathbb{L}_p(\Omega, \mathscr{A}, \mu) := \big\{\{f\}_\mu := f + \{0\}_\mu : f \in \mathcal{L}_p(\mu)\big\}.$$

For $\{f\}_\mu \in \mathbb{L}_p(\mu)$, define $\|\{f\}_\mu\|_{\mathbb{L}_p(\mu)} := \|f\|_{\mathcal{L}_p}$ for any $f \in \{f\}_\mu$. Also let $\mu(\{f\}_\mu) := \mu(f)$ if this expression is defined for f. Note that $\|\{f\}_\mu\|_{\mathbb{L}_p(\mu)}$ and $\mu(\{f\}_\mu)$ do not depend on the choice of the representative $f \in \{f\}_\mu$. Similarly, for $\{f\}_\mu, \{g\}_\mu \in \mathbb{L}_2(\mu)$ define

$$\langle \{f\}_\mu, \{g\}_\mu \rangle_{\mathbb{L}_2(\mu)} := \langle f, g \rangle_{\mathcal{L}_2(\mu)} = \mu(fg)$$

with $f \in \{f\}_\mu$ and $g \in \{g\}_\mu$. □

§02.50 **Lemma**. *Let $(\Omega, \mathscr{A}, \mu)$ be an arbitrary measure space and $f, g \in \overline{\mathcal{M}}(\mathscr{A})$.*

(i) *(Hölder's inequality) Let $s, r \in \overline{\mathbb{R}}_{\geqslant 1}$ with $\frac{1}{s} + \frac{1}{r} = 1$. Then $\mu(|fg|) \leqslant \|f\|_{\mathcal{L}_p}\|g\|_{\mathcal{L}_q}$.*
(Cauchy-Schwarz inequality) If $f, g \in \mathcal{L}_2$ then $|\langle f, g\rangle_{\mathcal{L}_2}| \leqslant \|f\|_{\mathcal{L}_2}\|g\|_{\mathcal{L}_2}$.

(ii) *If $\mu \in \mathfrak{M}_f(\mathscr{A})$ is finite, $s \in \overline{\mathbb{R}}_{>0}$ and $r \in (0, s)$. Then $\mu(\Omega)^{1/s}\|f\|_{\mathcal{L}_r(\mu)} \leqslant \mu(\Omega)^{1/r}\|f\|_{\mathcal{L}_s(\mu)}$ and hence $\mathcal{L}_s(\mu) \subseteq \mathcal{L}_r(\mu)$.*

(iii) *(Minkowski's inequality) For any $p \in \overline{\mathbb{R}}_{\geqslant 1}$, $\|f + g\|_{\mathcal{L}_p} \leqslant \|f\|_{\mathcal{L}_p} + \|g\|_{\mathcal{L}_p}$.*

(iv) *(Fischer-Riesz) For any $p \in \overline{\mathbb{R}}_{\geqslant 1}$, $(\mathbb{L}_p(\mu), \|\cdot\|_{\mathbb{L}_p(\mu)})$ is a Banach space. $(\mathbb{L}_2(\mu), \langle\cdot,\cdot\rangle_{\mathbb{L}_2(\mu)})$ is a real Hilbert space.*

§02.51 **Proof** of Lemma §02.50. For (i) and (iii) we refer to the lecture EWS or Bauer (1992, Satz 14.1/14.2, p.85/86). (ii) is shown in the lecture and (iv) can be found, for example, in Klenke (2008, Theorem 7.18, p.151) □

§02.52 **Remark**. Let $(V, \langle\cdot,\cdot\rangle)$ be a Hilbert space. Then the *Riesz-Fréchet representation theorem* states, that a map $F : V \to \mathbb{R}$ is continuous and linear if and only if there is an $f \in V$ with $F(x) = \langle f, x\rangle$ for all $x \in V$. The uniquely determined element $f \in V$ is called *representative* of F. In the next section we will need the representation theorem for the space $\mathcal{L}_2$, which unlike $\mathbb{L}_2$ is not a Hilbert space. The representation theorem still holds if V is a linear vector space and $\langle\cdot,\cdot\rangle$ is a complete positive semidefinite symmetric bilinear form (complete semi-inner product) (c.f. Klenke (2008) section 7.3). □

§02.53 **Lemma**. *The map $F : \mathcal{L}_2(\mu) \to \mathbb{R}$ is continuous and linear if and only if there is an $f \in \mathcal{L}_2(\mu)$ with $F(g) = \mu(gf)$ for all $g \in \mathcal{L}_2(\mu)$.*

§02.54 **Proof** of Lemma §02.53. we refer to Klenke (2008, Corollary 7.28, p.154) □

§03 Measures with density - Theorem of Radon-Nikodym

§03.01 **Definition**. Let $\nu, \mu \in \mathfrak{M}(\mathscr{A})$ be arbitrary measures on $(\Omega, \mathscr{A})$.

$\nu \ll \mu$: ν is called *absolutely continuous* with respect to μ, *μ-continuous*, or *dominated* by μ, if any μ-null set is also a ν-null set, that is, $\nu(A) = 0$ for all $A \in \mathscr{A}$ with $\mu(A) = 0$. The measures Maße μ and ν are called *equivalent* (symbolically $\mu \ll\gg \nu$), if $\nu \ll \mu$ and $\mu \ll \nu$.

$\mu \perp \nu$: μ is called *singular* to ν or *ν-singular*, if there exists a μ-null set $N \in \mathscr{A}$ such that $\nu(\Omega \setminus N) = 0$, or in equal $\nu = \mathbb{1}_N\nu$, that is, $\nu(A) = \nu(A \cap N)$ for all $A \in \mathscr{A}$. □

§03.02 **Remark**. Evidently, $\mu \perp \nu$ if and only if there are $\Omega_\mu, \Omega_\nu \in \mathscr{A}$ with $\Omega = \Omega_\mu \biguplus \Omega_\nu$ and $\mu(\Omega_\nu) = 0 = \nu(\Omega_\mu)$, and hence if and only if $\nu \perp \mu$. Consequently measures $\mu, \nu \in \mathfrak{M}(\mathscr{A})$ with $\mu \perp \nu$ are also called mutually singular. The condition $\nu = \mathbb{1}_N\nu$ means the support of the meassure ν is contained in $N \in \mathscr{A}$. Note that $\nu \ll \mu$ and $\nu \perp \mu$ imply together $\nu(N) = 0$, and hence $\nu = 0$. □

§03.03 **Lemma**. *Let $\nu, \mu \in \mathfrak{M}(\mathscr{A})$ be measures on $(\Omega, \mathscr{A})$. ν is called totally continuous with respect to μ if, for any $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta \in \mathbb{R}_{>0}$ such that $\nu(A) \leqslant \varepsilon$ for all $A \in \mathscr{A}$ with $\mu(A) \leqslant \delta$. If ν is totally continuous with respect to μ, then $\nu \ll \mu$. If $\nu \in \mathfrak{M}_e(\mathscr{A})$ is finite, then the converse also holds.*

§03.04 **Proof** of Lemma §03.03. is given in the lecture. □

Reminder. For measures $\mu, \nu \in \mathfrak{M}(\mathscr{A})$ we write $\nu \leqslant \mu$ if $\nu(A) \leqslant \mu(A)$ for all $A \in \mathscr{A}$. □

§03.05 **Lemma**. *Let $\nu, \mu \in \mathfrak{M}_f(\mathscr{A})$ be finite measures with $\nu \leqslant \mu$, then there exists $h \in \mathcal{M}_{[0,1]}(\mathscr{A})$ such that $\nu = h\mu$.*

§03.06 **Proof** of Lemma §03.05. is given in the lecture. □

§03.07 **Theorem of Radon-Nikodym**. *Let $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ be a σ-finite measure and let $\nu \in \mathfrak{M}(\mathscr{A})$ be a μ-continuous measure, i.e. $\nu \ll \mu$. Then ν has a density $\mathbb{f} = \mathrm{d}\nu/\mathrm{d}\mu \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ with respect to μ, that is, $\nu = \mathbb{f}\mu$.*

§03.08 **Proof** of Theorem §03.07. is given in the lecture. □

§03.09 **Remark**. Let $\mu, \nu \in \mathfrak{M}_\sigma(\mathscr{A})$ be σ-finite measures with $\nu \ll \mu$ and let $\mathbb{f} = \mathrm{d}\nu/\mathrm{d}\mu \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ be a μ-density of ν. Then Theorem §03.07 implies directly the usual *chain rules*:

(a) If $g \in \overline{\mathcal{M}}(\mathscr{A})$ is ν-quasiintegrable, then $\nu(g\mathbb{1}_A) = \mu(g\mathbb{f}\mathbb{1}_A)$ for all $A \in \mathscr{A}$.

(b) If $\rho \in \mathfrak{M}_\sigma(\mathscr{A})$ is a σ-finite measure with $\rho \ll \nu \ll \mu$ then $\frac{\mathrm{d}\rho}{\mathrm{d}\mu} = \frac{\mathrm{d}\rho}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}\mu}$ μ-a.e..

(c) If $h \in \mathcal{M}_{[0,1]}(\mathscr{A})$ with $h = \frac{\mathrm{d}\nu}{\mathrm{d}(\nu+\mu)}$ μ-a.e. then $\frac{\mathrm{d}\nu}{\mathrm{d}\mu} = \frac{h}{1-h}$ μ-a.e.. □

§03.10 **Example**.

(a) *Continuous probability measures* on $(\mathbb{R}^k, \mathscr{B}^k)$ as studied in the lecture EWS are probability measures dominated by the Lebesgue measure λ^k with corresponding (Radon-Nikodym-) density.

(b) *Discret probability measures* on a countable set Ω introduced in the lecture EWS are probability measures dominated by the counting measure ζ_Ω and the mass function corresponds to the (Radon-Nikodym-) density. Similarly, if $\Omega \subseteq \mathbb{R}$ then the discrete measure $\mu \in \mathfrak{M}_\sigma(\mathscr{B})$ with mass function $\mathbb{p}$ as in Example §01.18 (c) is absolutely continuous with respect to the counting measure $\zeta_\Omega \in \mathfrak{M}_\sigma(\mathscr{B})$ with (Radon-Nikodym-) density $\mathbb{p}$. □

§03.11 **Lebesgue's decomposition theorem**. *Let $\mu, \nu \in \mathfrak{M}_\sigma(\mathscr{A})$ be σ-finite measures on $(\Omega, \mathscr{A})$. Then there exists a unique decomposition $\nu = \nu_a + \nu_s$ of ν into two measures $\nu_a, \nu_s \in \mathfrak{M}(\mathscr{A})$ such* *that $\nu_a \ll \mu$ and $\nu_s \perp \mu$ is the μ-continuous, respectively the μ-singular part of ν. Moreover, $\nu_a, \nu_s \in \mathfrak{M}_\sigma(\mathscr{A})$ are σ-finite, and $\nu_a, \nu_s \in \mathfrak{M}_f(\mathscr{A})$ are finite if and only if $\nu \in \mathfrak{M}_f(\mathscr{A})$ is finite. ν_a has a μ-density $\mathrm{d}\nu_a/\mathrm{d}\mu \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ with $\mathrm{d}\nu_a/\mathrm{d}\mu \in \mathbb{R}_{\geqslant 0}$ μ-a.e.*

§03.12 **Proof** of Theorem §03.11. is given in the lecture. □

§03.13 **Remark**. If $\mathbb{f} = \mathrm{d}\nu_a/\mathrm{d}\mu \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ is a μ-density of ν_a as in Theorem §03.11 then the positive real function $\tilde{\mathbb{f}} := \mathbb{f}\mathbb{1}_{\{\mathbb{f}\in\mathbb{R}_{\geqslant 0}\}} \in \mathcal{M}_{\geqslant 0}(\mathscr{A})$ is also a μ-density of ν_a, since $\mathbb{f} = \tilde{\mathbb{f}}$ μ-a.e. In other words $\tilde{\mathbb{f}} \in \mathcal{M}_{\geqslant 0}(\mathscr{A})$ is also a version of the Radon-Nikodym density of ν_a with respect to μ. Consequently, without loss of generality we chose here and subsequently a positive real version of the Radon-Nikodym density. Furthermore, given $\mathbb{f} = \mathrm{d}\nu_a/\mathrm{d}\mu \in \mathcal{M}_{\geqslant 0}(\mathscr{A})$ let us define a numerical function $\mathrm{L} := \mathbb{f}\mathbb{1}_{N^c} + \infty\mathbb{1}_N \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ with $\mu(N) = 0 = \nu_s(N^c)$ where $\{\mathrm{L} = \infty\} = N$ and the Lebesgue decomposition writes $\nu = \mathrm{L}\mu + \mathbb{1}_{\{\mathrm{L}=\infty\}}\nu$, i.e. for all $A \in \mathscr{A}$ we have $\nu(A) = \mu(\mathbb{1}_A\mathrm{L}) + \nu(A \cap \{\mathrm{L} = \infty\})$. □

§03.14 **Definition**. Let $\nu, \mu \in \mathfrak{M}_\sigma(\mathscr{A})$ be two σ-finite measures on $(\Omega, \mathscr{A})$, where $\nu \ll \mu$ does not necessarily hold. Any positive numerical function $\mathrm{L} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ satisfying

$$\mu(\mathrm{L} = \infty) = 0 \text{ and } \nu = \mathrm{L}\mu + \mathbb{1}_{\{\mathrm{L}=\infty\}}\nu \tag{03.01}$$

is called *density ratio* of ν with respect to μ, or *μ-density ratio* of ν. □

§03.15 **Lemma**. *Let $\nu, \mu \in \mathfrak{M}_\sigma(\mathscr{A})$ be two σ-finite measures. Then the μ-density ratio $\mathrm{L} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$ of ν is unique up to $(\nu + \mu)$-a.e. equivalence.*

§03.16 **Proof** of Lemma §03.15. is given in the lecture. □

Alternative formulation of the theorem of Radon-Nikodym

§03.17 **Definition**. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space and let $\mathcal{F} \subseteq \overline{\mathcal{M}}(\mathscr{A})$ be a class of numerical functions. A function $g \in \overline{\mathcal{M}}(\mathscr{A})$ is called a *μ-essential supremum* over $\mathcal{F}$, symbolically $g = \mu\text{-ess}\sup_{f\in\mathcal{F}} f$, if (a) $f \leqslant g$ μ-a.e. for all $f \in \mathcal{F}$, and (b) if $h \in \overline{\mathcal{M}}(\mathscr{A})$ satisfies $f \leqslant h$ μ-a.e. for all $f \in \mathcal{F}$ then $g \leqslant h$ μ-a.e. □

§03.18 **Remark**. The μ-essential supremum can be seen as an extension of the usual concept of the supremum. If $\mathcal{F}$ is countable and $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ is σ-finite, then $g := \sup_{f\in\mathcal{F}} f$ satisfies the conditions §03.17 (a) and (b), and hence $\sup_{f\in\mathcal{F}} f = \mu\text{-ess}\sup_{f\in\mathcal{F}} f$ μ-a.e. In contrast, if for example $\mathcal{F} = \{\mathbb{1}_{\{x\}}, x \in B\}$ with uncountable $B \in \mathscr{B}$ such that $\lambda(B) \in \mathbb{R}_{>0}$, then the λ-essential supremum and the usual supremum differ. Precisely, $\sup_{f\in\mathcal{F}} f = \mathbb{1}_B \neq 0 = \lambda\text{-ess}\sup_{f\in\mathcal{F}} f$. □

§03.19 **Lemma**. *Let $\mu \in \mathfrak{M}_\sigma(\mathscr{A})$ and $\mathcal{F} \subseteq \overline{\mathcal{M}}(\mathscr{A})$. Then:*

(i) *$g := \mu\text{-ess}\sup_{f\in\mathcal{F}} f$ exists and it is μ-a.e. uniquely determined, that is, if $g \in \overline{\mathcal{M}}(\mathscr{A})$ is a solution of Definition §03.17 (a) and (b) then also $\widetilde{g} \in \overline{\mathcal{M}}(\mathscr{A})$ with $\mu(\{g \neq \widetilde{g}\}) = 0$.*

(ii) *There exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ with $g = \sup_{n\in\mathbb{N}} f_n$ μ-a.e.*

(iii) *If $\mathcal{F}$ is increasing filtered (for all $h, k \in \mathcal{F}$ exists $f \in \mathcal{F}$ with $f \geqslant h \vee k$), then there exists an isotone sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ with $f_n \uparrow g$ μ-a.e..*

§03.20 **Proof** of Lemma §03.19. We refer to Witting (1985, Satz 1.102, S.105). □

§03.21 **Lemma**. *Let $\mu, \nu \in \mathfrak{M}_f(\mathscr{A})$ be finite and mutually not singular measures on $(\Omega, \mathscr{A})$. Then there is $\Omega_\circ \in \mathscr{A}$ with $\mu(\Omega_\circ) \in \mathbb{R}_{>0}$ and $\varepsilon \in \mathbb{R}_{>0}$ with $\varepsilon\mathbb{1}_{\Omega_\circ}\mu \leqslant \mathbb{1}_{\Omega_\circ}\nu$.*

§03.22 **Proof** of Lemma §03.21. The claim is shown in Klenke (2020, Lemma 7.46, S.184) with help of the Hahn-decomposition for signed measures. An alternative proof of the claim is given in the proof of Bauer (1992, Satz 17.10, S.117) exploiting Bauer (1992, Lemma 17.9, S.114). □

§03.23 **Lemma**. *Let $\nu, \mu \in \mathfrak{M}_{\mathrm{f}}(\mathscr{A})$ be finite with $\nu \leqslant \mu$. Set $\mathcal{F} := \{f \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A}) : f\mu \leqslant \nu\}$ and $g := \mu\text{-ess sup}_{f \in \mathcal{F}} f$. Then $\nu = g\mu$, that is, g is a version of the μ-density of ν.*

§03.24 **Proof** of Lemma §03.23. is given in the lecture. □

§04 Measures on product spaces

§04|01 Finite product measures

§04.01 **Reminder**. Let $\mathcal{I}$ be an arbitrary nonempty index set and let $(\mathcal{S}_i, \mathscr{S}_i)$, $i \in \mathcal{I}$, be measurable spaces. The set $\mathcal{S}_{\mathcal{I}} := \times_{i \in \mathcal{I}} \mathcal{S}_i$ of all maps $(s_i)_{i \in \mathcal{I}} : \mathcal{I} \to \cup_{i \in \mathcal{I}} \mathcal{S}_i$ such that $s_i \in \mathcal{S}_i$ for all $i \in \mathcal{I}$ is called *product space* or *Cartesian product*. We identify the map $i \mapsto s_i$ and the family $(s_i)_{i \in \mathcal{I}}$. If $\mathcal{S}_i = \mathcal{S}$ for all $i \in \mathcal{I}$ then we write $\mathcal{S}^{\mathcal{I}} := \mathcal{S}_{\mathcal{I}}$, and in case $n := |\mathcal{I}| \in \mathbb{N}$ also $\mathcal{S}^n := \mathcal{S}^{\mathcal{I}}$ for short. For every $\mathcal{J} \subseteq \mathcal{I}$ the map $\Pi_{\mathcal{J}} : \mathcal{S}_{\mathcal{I}} \to \mathcal{S}_{\mathcal{J}}$ with $(s_i)_{i \in \mathcal{I}} \mapsto (s_j)_{j \in \mathcal{J}}$ is called *canonical projection* and in particular for $j \in \mathcal{I}$ the map $\Pi_j := \Pi_{\{j\}} : \mathcal{S}_{\mathcal{I}} \to \mathcal{S}_j$ with $(s_i)_{i \in \mathcal{I}} \mapsto s_j$ is called *coordinate map* such that $\times_{i \in \mathcal{I}} E_i = \bigcap_{i \in \mathcal{I}} \Pi_i^{-1}(E_i)$ for all $E_i \subseteq \mathcal{S}_i$ and $i \in \mathcal{I}$. □

§04.02 **Definition**. Let $\mathcal{I}$ be an arbitrary nonempty index set.

(a) Let $(\Omega, \mathscr{A})$ be a measurable space and for each $i \in \mathcal{I}$ let $\mathscr{A}_i \subseteq \mathscr{A}$ be a σ-algebra. The σ-algebra

$$\bigwedge_{i \in \mathcal{I}} \mathscr{A}_i := \bigcap_{i \in \mathcal{I}} \mathscr{A}_i \quad \text{and} \quad \bigvee_{i \in \mathcal{I}} \mathscr{A}_i := \sigma\big(\bigcup_{i \in \mathcal{I}} \mathscr{A}_i\big)$$

is respectively the *largest σ-algebra* on Ω, that belongs to all $\mathscr{A}_i$, $i \in \mathcal{I}$, and the *smallest σ-algebra* on Ω, that contains all $\mathscr{A}_i$, $i \in \mathcal{I}$.

(b) For each $i \in \mathcal{I}$ let $(\mathcal{S}_i, \mathscr{S}_i)$ be a measurable space. The *product-σ-algebra*

$$\mathscr{S}_{\mathcal{I}} := \bigotimes_{i \in \mathcal{I}} \mathscr{S}_i$$

is the smallest σ-algebra on the product space $\mathcal{S}_{\mathcal{I}} = \times_{i \in \mathcal{I}} \mathcal{S}_i$ such that for every $i \in \mathcal{I}$ the coordinate map $\Pi_i : \mathcal{S}_{\mathcal{I}} \to \mathcal{S}_i$ is measurable with respect to $\mathscr{S}_{\mathcal{I}}$-$\mathscr{S}_i$, i.e. $\Pi_i \in \mathcal{M}(\mathcal{S}_{\mathcal{I}}, \mathcal{S}_i)$; that is,

$$\mathscr{S}_{\mathcal{I}} = \bigotimes_{i \in \mathcal{I}} \mathscr{S}_i := \bigvee_{i \in \mathcal{I}} \sigma(\Pi_i) = \bigvee_{i \in \mathcal{I}} \Pi_i^{-1}(\mathscr{S}_i).$$

If $(\mathcal{S}_i, \mathscr{S}_i) = (\mathcal{S}, \mathscr{S})$ for all $i \in \mathcal{I}$, then we also write $\mathscr{S}^{\mathcal{I}} := \mathscr{S}_{\mathcal{I}}$, and $\mathscr{S}^n := \mathscr{S}^{\mathcal{I}}$ in case $n := |\mathcal{I}| \in \mathbb{N}$. The family $(\Pi_i)_{i \in \mathcal{I}}$ is called the *canonical process* on $(\mathcal{S}_{\mathcal{I}}, \mathscr{S}_{\mathcal{I}})$. □

Consider now the situation of finitely many measure spaces $(\mathcal{S}_i, \mathscr{S}_i, \mu_i)$, $i \in [\![n]\!]$, where $n \in \mathbb{N}$.

§04.03 **Lemma**. *For every $i \in [\![n]\!]$ let $\mathscr{E}_i$ be a generator of the σ-algebra $\mathscr{S}_i$ on $\mathcal{S}_i$ and let $(\mathcal{E}_{ik})_{k \in \mathbb{N}}$ be a sequence in $\mathscr{E}_i$ such that $\mathcal{E}_{ik} \uparrow \mathcal{S}_i$. Then the product-$\sigma$-algebra $\mathscr{S}_{[\![n]\!]} = \bigotimes_{i \in [\![n]\!]} \mathscr{S}_i$ is generated by the class of sets $\big\{\times_{i \in [\![n]\!]} \mathcal{E}_i : \mathcal{E}_i \in \mathscr{E}_i, i \in [\![n]\!]\big\}$.*

§04.04 **Proof** of Lemma §04.03. is given in the lecture. □

§04.05 **Remark.** Let $\mathscr{S}_1 = \{\emptyset, \mathcal{S}_1\}$ and $\mathscr{E}_1 = \{\emptyset\}$. Let $\mathscr{E}_2 = \mathscr{S}_2$ be a σ-algebra on $\mathcal{S}_2$ containing at least 4 elements. Then the class of sets $\{\emptyset \times E \colon E \in \mathscr{E}_2\}$ does not generate the product-σ-algebra $\mathscr{S}_1 \otimes \mathscr{S}_2$. Consequently, the restrictive assumption on the generator in Lemma §04.03 cannot simply be dispensed with. On the other hand side by applying Lemma §04.03 the product-σ-Algebra $\mathscr{S}_{[\![n]\!]} = \bigotimes_{i\in[\![n]\!]} \mathscr{S}_i$ is generated by the class of sets $\{\bigtimes_{i\in[\![n]\!]} \mathcal{E}_i \colon \mathcal{E}_i \in \mathscr{S}_i, i \in [\![n]\!]\}$ □

§04.06 **Definition.** A measure $\mu_{[\![n]\!]} \in \mathfrak{M}(\mathscr{S}_{[\![n]\!]})$ on $(\mathcal{S}_{[\![n]\!]}, \mathscr{S}_{[\![n]\!]})$ is called *product measure* if

$$\mu_{[\![n]\!]}\big(\bigtimes_{i\in[\![n]\!]} \mathcal{E}_i\big) = \mu_{[\![n]\!]}\Big(\bigcap_{i\in[\![n]\!]} \Pi_i^{-1}(\mathcal{E}_i)\Big) = \prod_{i\in[\![n]\!]} \mu_i(\mathcal{E}_i) \quad \text{for} \quad \mathcal{E}_i \in \mathscr{S}_i,\ i \in [\![n]\!].$$

In this case we write $\bigotimes_{i\in[\![n]\!]} \mu_i := \mu_{[\![n]\!]}$. If $\mu_i = \mu$ for all $i \in [\![n]\!]$, then we write $\mu^n := \mu_{[\![n]\!]}$. □

§04.07 **Lemma** *(Uniqueness of finite product measures). For every $i \in [\![n]\!]$ let $\mathscr{E}_i$ be a $\cap$-closed generator of the σ-algebra $\mathscr{S}_i$ on $\mathcal{S}_i$ and let* (uC) *$(\mathcal{E}_{ik})_{k\in\mathbb{N}}$ be a sequence in $\mathscr{E}_i$ such that $\mu_i(\mathcal{E}_{ik}) \in \mathbb{R}_{\geqslant 0}$ for every $k \in \mathbb{N}$ and $\mathcal{E}_{ik} \uparrow \mathcal{S}_i$.Then there is at most one measure $\mu_{[\![n]\!]} \in \mathfrak{M}(\mathscr{S}_{[\![n]\!]})$ on $(\mathcal{S}_{[\![n]\!]}, \mathscr{S}_{[\![n]\!]})$ with*

$$\mu_{[\![n]\!]}\big(\bigtimes_{i\in[\![n]\!]} \mathcal{E}_i\big) = \prod_{i\in[\![n]\!]} \mu_i(\mathcal{E}_i) \quad \textit{for} \quad \mathcal{E}_i \in \mathscr{E}_i,\ i \in [\![n]\!].$$

§04.08 **Proof** of Lemma §04.07. is given in the lecture. □

§04.09 **Remark.** Under the assumptions of Lemma §04.07 follows immediately that for every $i \in [\![n]\!]$ the measure $\mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_i)$ is σ-finite. □

§04.10 **Notation.** For $i \in [\![2]\!]$ let $(\mathcal{S}_i, \mathscr{S}_i)$ be a measurable space. For all $\mathcal{E} \subseteq \mathcal{S}_1 \times \mathcal{S}_2$, $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ we write $\mathcal{E}_{s_1} := \{s_2 \in \mathcal{S}_2 \colon (s_1, s_2) \in \mathcal{E}\}$ and $\mathcal{E}^{s_2} := \{s_1 \in \mathcal{S}_1 \colon (s_1, s_2) \in \mathcal{E}\}$. □

§04.11 **Lemma.** *For all $\mathcal{E} \in \mathscr{S}_1 \otimes \mathscr{S}_2$, $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ we have $\mathcal{E}_{s_1} \in \mathscr{S}_2$ und $\mathcal{E}^{s_2} \in \mathscr{S}_1$.*

§04.12 **Proof** of Lemma §04.11. is given in the lecture. □

§04.13 **Remark.** Due to Lemma §04.11 $\mu_2(\mathcal{E}_{s_1})$ and $\mu_1(\mathcal{E}^{s_2})$ are well-defined for all $\mathcal{E} \in \mathscr{S}_1 \otimes \mathscr{S}_2$, $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$. □

§04.14 **Lemma.** *For $i \in [\![2]\!]$ let $\mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_i)$ be a σ-finite measure on $(\mathcal{S}_i, \mathscr{S}_i)$. Then, for all $\mathcal{E} \in \mathscr{S}_1 \otimes \mathscr{S}_2$, the map $\mu_2(\mathcal{E}_\bullet) : s_1 \mapsto \mu_2(\mathcal{E}_{s_1})$ and $\mu_1(\mathcal{E}^\bullet) : s_2 \mapsto \mu_1(\mathcal{E}^{s_2})$ defined on $\mathcal{S}_1$ respectively $\mathcal{S}_2$ is positive numerical, that is, $\mu_2(\mathcal{E}_\bullet) \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S}_1)$ and $\mu_1(\mathcal{E}^\bullet) \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S}_2)$.*

§04.15 **Proof** of Lemma §04.14. is given in the lecture. □

§04.16 **Theorem** *(Existence of a product measure). For $i \in [\![2]\!]$ let $\mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_i)$ be a σ-finite measure on $(\mathcal{S}_i, \mathscr{S}_i)$. Then there exists a unique product measure $\mu_{[\![2]\!]}$ on $(\mathcal{S}_{[\![2]\!]}, \mathscr{S}_{[\![2]\!]})$. Moreover, $\mu_{[\![2]\!]} \in \mathfrak{M}_\sigma(\mathscr{S}_{[\![2]\!]})$ is also σ-finite and $\mu_1\big(\mu_2(\mathcal{E}_\bullet)\big) = \mu_{[\![2]\!]}(\mathcal{E}) = \mu_2\big(\mu_1(\mathcal{E}^\bullet)\big)$ for all $\mathcal{E} \in \mathscr{S}_{[\![2]\!]}$.*

§04.17 **Proof** of Theorem §04.16. is given in the lecture. □

§04.18 **Remark.** The last statement can easily be extended to a finite product measure. It should be noted that the parentheses in the products can be arbitrarily rearranged. Formally we identify the product sets $\mathcal{S}_{[\![n-1]\!]} \times \mathcal{S}_n$ und $\mathcal{S}_{[\![n]\!]}$ as usual with help of the bijection $((s_i)_{i\in[\![n-1]\!]}, s_n) \mapsto (s_i)_{i\in[\![n]\!]}$. The agreed equality of the sets implies then directly the equality of the corresponding products of σ-algebras $\mathscr{S}_{[\![n-1]\!]} \otimes \mathscr{S}_n$ and $\mathscr{S}_{[\![n]\!]}$ and the associative property $(\bigotimes_{i\in[\![m]\!]} \mathscr{S}_i) \otimes (\bigotimes_{i\in[\![n-m]\!]} \mathscr{S}_{m+i}) = \bigotimes_{i\in[\![n]\!]} \mathscr{S}_i$ for $m \in [\![n-1]\!]$. □

§04.19 **Corollary** *(Existence of product measures)*. *For $i \in [\![n]\!]$ let $\mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_i)$ be a σ-finite measure on $(\mathcal{S}_i, \mathscr{S}_i)$. Then there exists a unique σ-finite product measure $\mu_{[\![n]\!]} \in \mathfrak{M}_\sigma(\mathscr{S}_{[\![n]\!]})$ on $(\mathcal{S}_{[\![n]\!]}, \mathscr{S}_{[\![n]\!]})$.*

§04.20 **Proof** of Corollary §04.19. is given in the lecture. □

§04.21 **Remark**. For measures that are not necessarily σ-finite, it is still possible to prove the existence, but no longer the uniqueness, of a product measure. □

§04|02 Projective family

§04.22 **Reminder**. If (Ω_i, τ_i), $i \in \mathcal{I}$, are topological spaces, then the product topology τ on $\Omega_\mathcal{I}$ is the coarsest topology with respect to which all coordinate maps $\Pi_i : \Omega_\mathcal{I} \to \Omega_i$ are continuous. □

§04.23 **Lemma**. *Let $\mathcal{I}$ be countable, for every $i \in \mathcal{I}$ let $\mathcal{S}_i$ be a separable, complete metric space (Polish) with Borel σ-algebra $\mathscr{B}_i := \mathscr{B}_{\mathcal{S}_i}$ and let $\mathscr{B}_{\mathcal{S}_\mathcal{I}}$ be the Borel σ-algebra with respect to the product topology on $\mathcal{S}_\mathcal{I} = \times_{i\in\mathcal{I}} \mathcal{S}_i$. Then $\mathcal{S}_\mathcal{I}$ is Polish and $\mathscr{B}_{\mathcal{S}_\mathcal{I}} = \mathscr{B}_\mathcal{I} = \bigotimes_{i\in\mathcal{I}} \mathscr{B}_i$. In particular, $\mathscr{B}_{\mathbb{R}^n} = \mathscr{B}^n$ for $n \in \mathbb{N}$.*

§04.24 **Proof** of Lemma §04.23. We refer to Klenke (2008, Theorem 14.8, p.273) or Bauer (1992, Theorem 22.1, p.151). □

§04.25 **Definition**. Let $\mathcal{I}$ be an arbitrary nonempty index set and for any $\mathcal{J} \subseteq \mathcal{I}$ let $\Pi_\mathcal{J}$ be the canonical projection on $(\mathcal{S}_\mathcal{I}, \mathscr{S}_\mathcal{I})$. For any $\mathcal{E} \in \mathscr{S}_\mathcal{J}$, $\Pi_\mathcal{J}^{-1}(\mathcal{E}) \in \mathscr{S}_\mathcal{I}$ is called a *cylinder set* with base $\mathcal{J}$. The set of such cylinder sets is denoted by $\mathcal{Z}_\mathcal{J} := \{\Pi_\mathcal{J}^{-1}(\mathcal{E}) : \mathcal{E} \in \mathscr{S}_\mathcal{J}\} \subseteq \mathscr{S}_\mathcal{I}$. In particular, if $\mathcal{E}_\mathcal{J} = \times_{j\in\mathcal{J}} \mathcal{E}_j \in \mathscr{S}_\mathcal{J}$, then $\Pi_\mathcal{J}^{-1}(\mathcal{E}) \in \mathscr{S}_\mathcal{I}$ is called a *rectangular cylinder* with base $\mathcal{J}$. The set of such rectangular cylinders will be denoted by $\mathcal{Z}_\mathcal{J}^R := \{\Pi_\mathcal{J}^{-1}(\mathcal{E}_\mathcal{J}) : \mathcal{E}_\mathcal{J} = \times_{j\in\mathcal{J}} \mathcal{E}_j \in \mathscr{S}_\mathcal{J}\} \subseteq \mathscr{S}_\mathcal{I}$. For every $i \in \mathcal{I}$ let $\mathscr{E}_i \subseteq \mathscr{S}_i$. The set of rectangular cylinders for which in addition $\mathcal{E}_j \in \mathscr{E}_j$ for all $j \in \mathcal{J}$ holds will be denoted by $\mathcal{Z}_\mathcal{J}^{\mathscr{E},R} := \{\Pi_\mathcal{J}^{-1}(\mathcal{E}_\mathcal{J}) : \mathcal{E}_\mathcal{J} = \times_{j\in\mathcal{J}} \mathcal{E}_j, \mathcal{E}_j \in \mathscr{E}_j, j \in \mathcal{J}\} \subseteq \mathscr{S}_\mathcal{I}$. Write $\mathcal{Z} := \bigcup\{\mathcal{Z}_\mathcal{J} : \mathcal{J} \subseteq \mathcal{I} \text{ finite}\}$ and similarly define $\mathcal{Z}^R$ and $\mathcal{Z}^{\mathscr{E},R}$. □

§04.26 **Remark**. Every $\mathcal{Z}_\mathcal{J}$ is a σ-algebra, and $\mathcal{Z}$ is a algebra where $\mathscr{S}_\mathcal{I} = \sigma(\mathcal{Z})$. Moreover, if every $\mathscr{E}_i$ is $\cap$-closed, then $\mathcal{Z}^{\mathscr{E},R}$ is also $\cap$-closed (Exercise). □

§04.27 **Lemma**. *For any $i \in \mathcal{I}$ let $\mathscr{E}_i \subseteq \mathscr{S}_i$ be a generator of $\mathscr{S}_i$.*

(i) *$\mathscr{S}_\mathcal{J} = \sigma(\times_{j\in\mathcal{J}} \mathcal{E}_j : \mathcal{E}_j \in \mathscr{E}_j, j \in \mathcal{J})$ for every finite $\mathcal{J} \subseteq \mathcal{I}$.*

(ii) *$\mathscr{S}_\mathcal{I} = \sigma(\mathcal{Z}^R) = \sigma(\mathcal{Z}^{\mathscr{E},R})$.*

(iii) *Let (A1) $\mu \in \mathfrak{M}_\sigma(\mathscr{S}_\mathcal{I})$ be a σ-finite measure on $(\mathcal{S}_\mathcal{I}, \mathscr{S}_\mathcal{I})$, assume (A2) every $\mathscr{E}_i$ is $\cap$-closed, and (A3) there is a sequence $(\mathcal{E}_n)_{n\in\mathbb{N}}$ in $\mathcal{Z}^{\mathscr{E},R}$ with $\mathcal{E}_n \uparrow \mathcal{S}_\mathcal{I}$ and $\mu(\mathcal{E}_n) \in \mathbb{R}_{\geqslant 0}$ for all $n \in \mathbb{N}$. Then μ is uniquely determined by the values $\mu(A)$ for all $A \in \mathcal{Z}^{\mathscr{E},R}$.*

§04.28 **Proof** of Lemma §04.27. Exercise. □

§04.29 **Comment**. The condition (A3) in Lemma §04.27 (iii) is fulfilled, if $\mu \in \mathfrak{M}_e(\mathscr{S}_\mathcal{I})$ is finite and $\mathcal{S}_i \in \mathscr{E}_i$ for every $i \in \mathcal{I}$ (compare Lemma §01.28). □

§04.30 **Notation**. For $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$ the map $\Pi_\mathcal{J}^\mathcal{K} : \mathcal{S}_\mathcal{K} \to \mathcal{S}_\mathcal{J}$ with $(s_k)_{k\in\mathcal{K}} \mapsto (s_j)_{j\in\mathcal{J}}$ is called *canonical projection*, where evidently $\Pi_\mathcal{J} = \Pi_\mathcal{J}^\mathcal{I}$. □

§04.31 **Definition**. For every finite $\mathcal{J} \subseteq \mathcal{I}$ let $\mathbb{P}_\mathcal{J} \in \mathcal{W}(\mathscr{S}_\mathcal{J})$ be a probability measure on $(\mathcal{S}_\mathcal{J}, \mathscr{S}_\mathcal{J})$. The familiy $\{\mathbb{P}_\mathcal{J} : \mathcal{J} \subseteq \mathcal{I} \text{ finite}\}$ is called *projective* or *consistent* if $\mathbb{P}_\mathcal{J} = \mathbb{P}_\mathcal{K} \circ (\Pi_\mathcal{J}^\mathcal{K})^{-1}$ for all finite $\mathcal{J} \subseteq \mathcal{K} \subseteq \mathcal{I}$. □

§04.32 **Remark**. Let $\mathbb{P} \in \mathcal{W}(\mathscr{S}_\mathcal{J})$ be a probability measure on $(\mathcal{S}_\mathcal{J}, \mathscr{S}_\mathcal{J})$. Since $\Pi_\mathcal{J} = \Pi^\mathcal{K}_\mathcal{J} \circ \Pi_\mathcal{K}$, the family $\{\mathbb{P}_\mathcal{J} := \mathbb{P} \circ \Pi_\mathcal{J} : \mathcal{J} \subseteq \mathcal{I} \text{ finite}\}$ is consistent. Thus, consistency is a necessary condition for the existence of a measure $\mathbb{P}$ on the product space with $\mathbb{P}_\mathcal{J} := \mathbb{P} \circ \Pi_\mathcal{J}$. If all the measurable spaces are Polish, spaces then this condition is also sufficient. □

§04.33 **Kolmogorov's extension theorem**. *Let $\mathcal{I}$ be an arbitrary nonempty index set and let $\mathcal{S}_i$ be a separable and complete metric space (Polish) with Borel σ algebra $\mathscr{B}_i := \mathscr{B}_{\mathcal{S}_i}$ for all $i \in \mathcal{I}$. Let $\{\mathbb{P}_\mathcal{J} : \mathcal{J} \subseteq \mathcal{I} \text{ finite}\}$ be a consistent family of probability measures. Then there exists a unique probability measure $\mathbb{P} \in \mathcal{W}(\mathscr{B}_\mathcal{I})$ on $(\mathcal{S}_\mathcal{I}, \mathscr{B}_\mathcal{I})$ with $\mathbb{P}_\mathcal{J} = \mathbb{P} \circ \Pi_\mathcal{J}^{-1}$ for all finite $\mathcal{J} \subseteq \mathcal{I}$. $\mathbb{P}$ is called* projective limit.

§04.34 **Proof** of Theorem §04.33. We refer to Klenke (2008, Theorem 14.36, p. 287) □

§04.35 **Definition**. Let $\mathbb{P}_i \in \mathcal{W}(\mathscr{S}_i)$ be a probability measure on $(\mathcal{S}_i, \mathscr{S}_i)$ for all $i \in \mathcal{I}$. A probability measure $\mathbb{P}_\mathcal{I} \in \mathcal{W}(\mathscr{S}_\mathcal{I})$ on $(\mathcal{S}_\mathcal{I}, \mathscr{S}_\mathcal{I})$ is called *product measure* of the $\mathbb{P}_i$, $i \in \mathcal{I}$, if

$$\mathbb{P}_\mathcal{I}\big(\times_{j\in\mathcal{J}} \mathcal{E}_i\big) = \mathbb{P}_\mathcal{I}\big(\bigcap_{j\in\mathcal{J}} \Pi_j^{-1}(\mathcal{E}_j)\big) = \prod_{j\in\mathcal{J}} \mathbb{P}_j(\mathcal{E}_j) \quad \text{for} \quad \mathcal{E}_j \in \mathscr{S}_j,\ j \in \mathcal{J} \subseteq \mathcal{I} \text{ finite.}$$

In this case we write $\bigotimes_{i\in\mathcal{I}} \mathbb{P}_i := \mathbb{P}_\mathcal{I}$. If $\mathbb{P}_i = \mathbb{P}$ for all $i \in \mathcal{I}$ then $\mathbb{P}^\mathcal{I} := \mathbb{P}_\mathcal{I}$ and $\mathbb{P}^n := \mathbb{P}_\mathcal{I}$ in case $n := |\mathcal{I}| \in \mathbb{N}$. □

§04.36 **Remark**. Let $\mathcal{I}$ be an arbitrary nonempty index set. For every $i \in \mathcal{I}$ let $\mathcal{S}_i$ be a separable and complete metric space (Polish) with Borel σ-algebra $\mathscr{B}_i := \mathscr{B}_{\mathcal{S}_i}$ and $\mathbb{P}_i \in \mathcal{W}(\mathscr{B}_i)$ be a probability measure on $(\mathcal{S}_i, \mathscr{B}_i)$. For every finite $\mathcal{J} \subseteq \mathcal{I}$ let $\mathbb{P}_\mathcal{J} := \bigotimes_{j\in\mathcal{J}} \mathbb{P}_j$ be the finite product measure of the $\mathbb{P}_j$, $j \in \mathcal{J}$. Evidently, the family $\{\mathbb{P}_\mathcal{J} : \mathcal{J} \subseteq \mathcal{I} \text{ finite}\}$ is *projective*. Making use of Theorem §04.33 there exists a unique product measure $\mathbb{P}_\mathcal{I} := \bigotimes_{i\in\mathcal{I}} \mathbb{P}_i \in \mathcal{W}(\mathscr{B}_\mathcal{I})$ on $(\mathcal{S}_\mathcal{I}, \mathscr{B}_\mathcal{I})$. Considering the canonical process $(\Pi_i)_{i\in\mathcal{I}}$ under $\mathbb{P}_\mathcal{I}$, all coordinate maps Π_i are independent, i.e. $\perp\!\!\!\perp_{i\in\mathcal{I}} \Pi_i$. □

§04|03 Integration with respect to product measures

§04.37 **Notation**. Let $h : \mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_3$ be a map. For all $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ we write $h_{s_1} : \mathcal{S}_2 \to \mathcal{S}_3$ with $s_2 \mapsto h_{s_1}(s_2) := h(s_1, s_2)$ and $h^{s_2} : \mathcal{S}_1 \to \mathcal{S}_3$ with $s_1 \mapsto h^{s_2}(s_1) := h(s_1, s_2)$. □

§04.38 **Lemma**. *For $i \in [\![3]\!]$, let $(\mathcal{S}_i, \mathscr{S}_i)$ be a measurable space. For all $h \in \mathcal{M}(\mathscr{S}_1 \otimes \mathscr{S}_2, \mathscr{S}_3)$, $s_1 \in \mathcal{S}_1$ and $s_2 \in \mathcal{S}_2$ we have $h_{s_1} \in \mathcal{M}(\mathscr{S}_2, \mathscr{S}_3)$ and $h^{s_2} \in \mathcal{M}(\mathscr{S}_1, \mathscr{S}_3)$.*

§04.39 **Proof** of Lemma §04.38. is given in the lecture. □

§04.40 **Theorem** *(Tonelli)*. *For $i \in [\![2]\!]$ let $\mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_i)$ be a σ-finite measure on $(\mathcal{S}_i, \mathscr{S}_i)$. Then, for every $h \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S}_1 \otimes \mathscr{S}_2)$ the map $\mu_1(h^\bullet) : s_2 \mapsto \mu_1(h^{s_2})$ and $\mu_2(h_\bullet) : s_1 \mapsto \mu_2(h_{s_1})$ defined on $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, is positive numerical, that is, $\mu_1(h^\bullet) \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S}_2)$ and $\mu_2(h_\bullet) \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S}_1)$. Moreover, it holds*

$$(\mu_1 \otimes \mu_2)(h) = \mu_2(\mu_1(h^\bullet)) = \int \mu_1(h^{s_2})\mu_2(\mathrm{d}s_2) = \int\int h(s_1, s_2)\mu_1(\mathrm{d}s_1)\mu_2(\mathrm{d}s_2)$$
$$= \int \mu_2(h_{s_1})\mu_1(\mathrm{d}s_1) = \mu_1(\mu_2(h_\bullet))$$

§04.41 **Proof** of Theorem §04.40. is given in the lecture. □

§04.42 **Definition**. Let $(\Omega, \mathscr{A}, \mu)$ be a measure space, $(\mathcal{S}, \mathscr{S})$ be a measurable space, and $N \in \mathscr{A}$ be an μ-null set. A function $h : N^c := \Omega \setminus N \to \mathcal{S}$ is called *μ-almost everywhere defined* and *$\mathscr{A}$-$\mathscr{S}$-measurable* if $h^{-1}(\mathscr{S}) \subseteq \mathscr{A}$ holds.

§04.43 **Remark**. If $h, g \in \overline{\mathcal{M}}(\mathscr{A})$ are μ-almost everywhere finite, then the function $g - h$ is μ-almost everywhere defined and $\mathscr{A}$-$\overline{\mathscr{B}}$-measurable. This holds in particular if g and h are μ-integrable. Now, if f is $\overline{\mathbb{R}}$-valued, μ-almost everywhere defined with μ-null set N and $\mathscr{A}$-$\overline{\mathscr{B}}$-measurable, then we can define $\tilde{f}(\omega) := 0$ for $\omega \in N$ and otherwise $\tilde{f}(\omega) := f(\omega)$. Then $\tilde{f} \in \overline{\mathcal{M}}(\mathscr{A})$ is numerical. If $\tilde{f}$ is furthermore μ-integrable, then we define for f the integral $\mu(f) = \int f \, d\mu := \mu(\tilde{f})$. □

§04.44 **Corollary** *(Fubini's theorem)*. *Let $(\mathcal{S}_i, \mathscr{S}_i, \mu_i)$, $i \in [\![2]\!]$, be σ-finite measure spaces and $h \in \mathcal{L}_1(\mu_1 \otimes \mu_2)$. Then $\mu_2(h_\bullet) : s_1 \mapsto \mu_2(h_{s_1})$ is μ_1-almost everywhere defined and $\mathscr{S}_1$-$\overline{\mathscr{B}}$-measurable, and $\mu_1(h^\bullet) : s_2 \mapsto \mu_1(h^{s_2})$ is μ_2-almost everywhere defined and $\mathscr{S}_2$-$\overline{\mathscr{B}}$-measurable. It holds that*

$$\mu_2(\mu_1(h^\bullet)) = \int \mu_1(h^{s_2}) \mu_2(ds_2) = (\mu_1 \otimes \mu_2)(h) = \int \mu_2(h_{s_1}) \mu_1(ds_1) = \mu_1(\mu_2(h_\bullet)).$$

§04.45 **Proof** of Corollary §04.44. is given in the lecture. □

§04.46 **Remark**. The last statements can be easily extended to finite product measures, as in Remark §04.18. □

§04.47 **Theorem**. *For each $i \in [\![n]\!]$, let $(\mathcal{S}_i, \mathscr{S}_i, \mu_i)$ be a σ-finite measure space, $\mathfrak{f}_i \in \mathcal{M}_{\geqslant 0}(\mathscr{A}_i)$, and $\nu_i := \mathfrak{f}_i \mu_i$. Then the product measure $\nu_{[\![n]\!]} = \prod_{i \in [\![n]\!]} \nu_i \in \mathfrak{M}_\sigma(\mathscr{S}_{[\![n]\!]})$ is σ-finite and absolutely continuous with respect to the product measure $\mu_{[\![n]\!]} = \prod_{i \in [\![n]\!]} \mu_i \in \mathfrak{M}_\sigma(\mathscr{S}_{[\![n]\!]})$ with product density $\prod_{i \in [\![n]\!]} \mathfrak{f}_i \in \mathcal{M}_{\geqslant 0}(\mathscr{A}_{[\![n]\!]})$, meaning $\nu_{[\![n]\!]} = \left(\prod_{i \in [\![n]\!]} \mathfrak{f}_i\right) \mu_{[\![n]\!]}$.*

§04.48 **Proof** of Theorem §04.47. is given in the lecture. □

§04.49 **Reminder**. Now, let $\nu = \mathbb{P}_0$ and $\mu = \mathbb{P}_1$ be probability measures on $(\mathcal{S}, \mathscr{S})$, where it is not necessarily the case that $\mathbb{P}_0 \ll \mathbb{P}_1$. Then any positive, measurable function $\mathrm{L} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S})$ with $\mathbb{P}_0 = \mathrm{L}\mathbb{P}_1 + \mathbb{1}_{\mathrm{L}=\infty}\mathbb{P}_0$ and $\mathbb{P}_1(\mathrm{L} \in \mathbb{R}_{\geqslant 0}) = 1$ is is a $\mathbb{P}_1$-density ratio of $\mathbb{P}_0$ (cf. Definition §03.14). Let $\mu \in \mathfrak{M}_\sigma(\mathscr{S})$ denote a σ-finite measure such that $\mathbb{P}_i \ll \mu$, $i \in [\![2]\!]$, (for example, the finite measure $\mu = \mathbb{P}_0 + \mathbb{P}_1$), and let $\mathfrak{f}_i \in \mathcal{M}_{\geqslant 0}(\mathscr{S})$ be a μ-density of $\mathbb{P}_i$, $i \in [\![2]\!]$. Then

$$\mathrm{L}_\star := \frac{\mathfrak{f}_0}{\mathfrak{f}_1} \mathbb{1}_{\{\mathfrak{f}_1 \in \mathbb{R}_{>0}\}} + \infty \mathbb{1}_{\{\mathfrak{f}_1 = 0\} \cap \{\mathfrak{f}_0 \in \mathbb{R}_{>0}\}} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S})$$

is a specific choice of the $\mathbb{P}_1$-density ratio of $\mathbb{P}_0$. We note that a $\mathbb{P}_0$-density ratio of $\mathbb{P}_1$ is given by

$$\mathrm{L}_\star^{-1} = \frac{\mathfrak{f}_1}{\mathfrak{f}_0} \mathbb{1}_{\{\mathfrak{f}_0 \in \mathbb{R}_{>0}\}} + \infty \mathbb{1}_{\{\mathfrak{f}_0 = 0\} \cap \{\mathfrak{f}_1 \in \mathbb{R}_{>0}\}} \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S})$$

In the special case where $\mathbb{P}_0 \ll \mathbb{P}_1$, the $\mathbb{P}_1$-density ratio of $\mathbb{P}_0$ is a $\mathbb{P}_1$-density of $\mathbb{P}_0$ and is $\mathbb{P}_1$-determined. □

§04.50 **Lemma**. *For each $i \in [\![n]\!]$, let $\mathbb{P}_{0|i}, \mathbb{P}_{1|i} \in \mathcal{W}(\mathscr{S}_i)$ be probability measures on $(\mathcal{S}_i, \mathscr{S}_i)$ with $\mathbb{P}_{1|i}$-density ratio L_i of $\mathbb{P}_{0|i}$. Then the product $\mathrm{L} := \prod_{i \in [\![n]\!]} \mathrm{L}_i$ is a density ratio of $\mathbb{P}_0 := \bigotimes_{i \in [\![n]\!]} \mathbb{P}_{0|i}$ with respect to $\mathbb{P}_1 := \bigotimes_{i \in [\![n]\!]} \mathbb{P}_{1|i}$.*

§04.51 **Proof** of Lemma §04.50. is given in the lecture. □

§04|04 Integration with respect to transition kernel

§04.52 **Definition**. Let $(\Omega, \mathscr{A})$ and $(\mathcal{S}, \mathscr{S})$ be two measurable spaces. A map $\kappa : \Omega \times \mathscr{S} \to \overline{\mathbb{R}}_{\geqslant 0}$ is called a (σ-)finite *transition kernel* from $(\Omega, \mathscr{A})$ to $(\mathcal{S}, \mathscr{S})$ if it satisfies the following two conditions:

(tK1) For all $\omega \in \Omega$, $\kappa_\omega : \mathscr{S} \to \overline{\mathbb{R}}_{\geqslant 0}$ with $S \mapsto \kappa_\omega(S) := \kappa(\omega, S)$ is a (σ-)finite measure on $(\mathcal{X}, \mathscr{X})$, i.e. $\kappa_\omega \in \mathfrak{M}_e(\mathscr{S})$ (respectively $\kappa_\omega \in \mathfrak{M}_\sigma(\mathscr{S})$.

(tK2) For all $S \in \mathscr{S}$, $\kappa^S : \Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ with $\omega \mapsto \kappa^S(\omega) := \kappa(\omega, S)$ is positive, numerical and $\mathscr{S}$-measurable, i.e. $\kappa^S \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{S})$.

If for every $\omega \in \Omega$, the measure in (tK1) is a probability measure, $\kappa_\omega \in \mathcal{W}(\mathscr{S})$, then κ is called a *Markov kernel*. □

§04.53 **Remark**. It suffices to require condition (tK2) only for sets from a $\cap$-closed generator $\mathscr{E}$ of $\mathscr{S}$, which contains $\mathcal{S}$ or a sequence $(\mathcal{E}_n)_{n \in \mathbb{N}}$ of sets such that $\mathcal{E}_n \uparrow \mathcal{S}$. Then $\mathscr{D} := \{S \in \mathscr{S} : \kappa^S \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})\}$ is a Dynkin system (exercise) with $\mathscr{E} \subseteq \mathscr{D} \subseteq \mathscr{S}$, and from the π-λ-Theorem §01.11, it follows that $\mathscr{D} = \sigma(\mathscr{E}) = \mathscr{S}$. □

§04.54 **Lemma**. *Let κ be a finite transition kernel from $(\Omega, \mathscr{A})$ to $(\mathcal{S}, \mathscr{S})$, and let $h \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A} \otimes \mathscr{S})$ be positive numerical. Then the function $\kappa_\bullet(h_\bullet) : \Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ defined by*

$$\omega \mapsto \kappa_\omega(h_\omega) = \int h_\omega \, \mathrm{d}\kappa_\omega$$

is well-defined and belongs to $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$.

§04.55 **Proof** of Lemma §04.54. is given in the lecture. □

§04.56 **Notation**. For $\mathbb{1}_A \in \mathcal{M}_{\geqslant 0}(\mathscr{A} \otimes \mathscr{S})$, that is, $A \in \mathscr{A} \otimes \mathscr{S}$, according to Lemma §04.54, the function $\kappa_\bullet(A_\bullet) = \kappa_\bullet\big((\mathbb{1}_A)_\bullet\big) : \Omega \to \overline{\mathbb{R}}_{\geqslant 0}$ defined by

$$\omega \mapsto \kappa_\omega(A_\omega) = \kappa_\omega\big((\mathbb{1}_A)_\omega\big) = \int \mathbb{1}_A(\omega, s)\kappa_\omega(\mathrm{d}s)$$

is well-defined and belongs to $\overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A})$. □

§04.57 **Lemma**. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space, $(\mathcal{S}, \mathscr{S})$ be a measurable space, and κ be a finite transition kernel from $(\Omega, \mathscr{A})$ to $(\mathcal{S}, \mathscr{S})$. Then there exists a uniquely determined σ-finite measure $\mu \odot \kappa \in \mathfrak{M}_\sigma(\mathscr{A} \otimes \mathscr{S})$ on the product space $(\Omega \times \mathcal{S}, \mathscr{A} \otimes \mathscr{S})$ such that*

$$(\mu \odot \kappa)(B) = \mu(\kappa_\bullet(B_\bullet)) \quad \text{for } B \in \mathscr{A} \otimes \mathscr{S},$$

where for all $A \in \mathscr{A}$ and $S \in \mathscr{S}$, we have

$$(\mu \odot \kappa)(A \times S) = \mu\big(\mathbb{1}_A \kappa^S\big) = \int_A \kappa^S \, \mathrm{d}\mu = \int_A \kappa(\omega, S)\mu(\mathrm{d}\omega).$$

If κ is a Markov kernel and μ is a probability measure, then $\mu \odot \kappa$ is a probability measure.

§04.58 **Proof** of Lemma §04.57. is given in the lecture. □

§04.59 **Theorem** *(Tonelli/Fubini for transition kernel)*. *Let $(\Omega, \mathscr{A}, \mu)$ be a finite measure space, $(\mathcal{S}, \mathscr{S})$ be a measurable space, and κ be a finite transition kernel from $(\Omega, \mathscr{A})$ to $(\mathcal{S}, \mathscr{S})$. If $h \in \overline{\mathcal{M}}_{\geqslant 0}(\mathscr{A} \otimes \mathscr{S})$*

or $h \in \mathcal{L}_1(\mu \odot \kappa)$ *then*

$$(\mu \odot \kappa)(h) = \mu(\kappa_\bullet(h_\bullet)) = \int \kappa_\omega(h_\omega)\mu(\mathrm{d}\omega) = \int \Big(\int h_\omega \,\mathrm{d}\kappa_\omega \Big)\mu(\mathrm{d}\omega)$$
$$= \int \int h(\omega, s)\kappa(\omega, \mathrm{d}s)\mu(\mathrm{d}\omega).$$

§04.60 **Proof** of Theorem §04.59. is given in the lecture. □

§04.61 **Notation.** Consider a probability space $(\Omega, \mathscr{A}, \mathbb{P})$, a measurable space $(\mathcal{S}, \mathscr{S})$, and a Markov kernel κ from $(\Omega, \mathscr{A})$ to $(\mathcal{S}, \mathscr{S})$. Due to Lemma §04.57, $\mathbb{P} \odot \kappa \in \mathcal{W}(\mathscr{A} \otimes \mathscr{S})$ is a uniquely determined probability measure on $(\Omega \times \mathcal{S}, \mathscr{A} \otimes \mathscr{S})$. Then we denote by

$$(\kappa\mathbb{P})(S) := \mathbb{P}\big(\kappa^S\big) = \int \kappa^S \,\mathrm{d}\mathbb{P} = \int \kappa(\omega, S)\mathbb{P}(\mathrm{d}\omega), \quad \text{for } S \in \mathscr{S}$$

the marginal distribution $\kappa\mathbb{P} \in \mathcal{W}(\mathscr{S})$ on $(\mathcal{S}, \mathscr{S})$ induced by $\mathbb{P} \odot \kappa \in \mathcal{W}(\mathscr{A} \otimes \mathscr{S})$. □

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